Analiza 2b (VAJE)

FOURIEROVE VRSTE

$f: [-\pi, \pi] \to \mathbb{R}$ zvezna, razen morda v končno mnogo točkah, kjer ima levo in desno limito
f je odvedljiva in v točkah nezveznosti obstajata levi in desni odvod

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$ ← RAZVOJ V FOURIEROVO VRSTO

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$FV(f)(x)$ konvergira po točkah proti $f(x)$, če je f zvezna v x, in proti $\frac{f(x-)+f(x+)}{2}$, če f ni zvezna v x.

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1. Dana je funkcija f s predpisom

$$f(x) = \begin{cases} x & ; \quad -\pi \le x \le 0 \\ 2x & ; \quad 0 < x \le \pi \end{cases}.$$

Razvij funkcijo f v Fourierovo vrsto na intervalu $[-\pi, \pi]$ in se prepričaj, da dobljena vrsta povsod konvergira proti funkciji $g(x)$. Nato skiciraj še graf funkcije $g(x)$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} 2x\,dx\right) = \frac{1}{2\pi}\left(-\frac{\pi^2}{2} + \pi^2\right) = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\cos(nx)\,dx + \int_{0}^{\pi} 2x\cos(nx)\,dx\right)$$

$$\int x\cos(nx) \underset{\text{PER PARTES}}{=} x\,\frac{1}{n}\sin(nx) - \int \frac{1}{n}\sin(nx)\,dx =$$

PER PARTES
$u = x \qquad dv = \cos(nx)\,dx$
$du = dx \qquad v = \frac{1}{n}\sin(nx)$

$$= \frac{x}{n}\sin(nx) - \frac{1}{n}\left(-\frac{1}{n}\cos(nx)\right) + C =$$

$$= \frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx) + C$$

$$\Rightarrow a_n = \frac{1}{\pi}\left(\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{-\pi}^{0} + 2\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{0}^{\pi}\right) =$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{1}{n^2}(-1)^n + 2\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)\right) =$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right) = \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\sin(nx)\,dx$$

$$\int x\sin(nx) \underset{\uparrow}{=} -\frac{x}{n}\cos(nx) + \int \frac{1}{n}\cos(nx)\,dx =$$

PER PARTES

$u = x \quad dv = \sin(nx)dx$

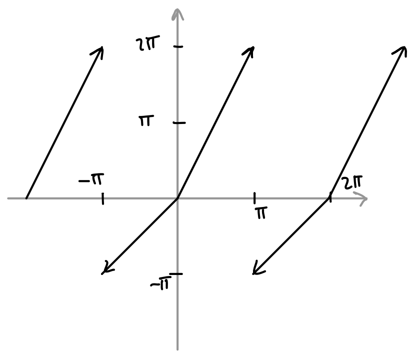
$$= -\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx) + C$$

$$\Rightarrow \underline{b_n} = \frac{1}{\pi}\left(\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{0} + 2\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{0}^{\pi}\right) =$$

$$= \underline{\frac{3}{n}(-1)^{n+1}}$$

$$\underline{FV(f)(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{\pi n^2}\cos(nx) + \frac{3}{n}(-1)^{n+1}\sin(nx)\right)}$$

Graf:



Opomba: Rabili smo $\underbrace{\int x\cos(nx)\,dx}_{I_1}$ in $\underbrace{\int x\sin(nx)\,dx}_{I_2}$

$\cos(nx) + i\sin(nx) = e^{inx}$

Združimo: $\int e^{inx}\,dx \underset{\uparrow}{=} \cdots = I_1 + I_2 + C$

PER PARTES

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2. Razvij funkcijo $f(x) = |x|$ v Fourierovo vrsto na intervalu $[-\pi, \pi]$ in seštej vrsto $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0}-x\,dx + \int_{0}^{\pi}x\,dx\right) = \frac{1}{2\pi}\cdot 2\int_{0}^{\pi}x\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi}x\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{0}^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right) =$$

$$= \underline{\frac{2((-1)^n-1)}{\pi n^2}}$$

$b_n = 0 \Leftarrow f(x)$ soda funkcija

$$\Rightarrow FV(f)(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\underbrace{\frac{2((-1)^n-1)}{\pi n^2}}_{\text{za sode } n \text{ dobimo } 0}\cos(nx) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi}\cos((2n-1)x)$$

$$FV(f)(0) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2\pi} = \frac{\pi}{2} - \frac{4}{\pi} S \quad \Big\} \to \frac{\pi}{2} - \frac{4}{\pi} S = 0$$

$$f(0) = 0$$

$$\Rightarrow \underline{S = \frac{\pi^2}{8}}$$

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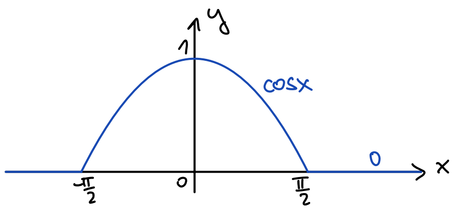
3. Razvij funkcijo $f(x) = \max(\cos x, 0)$ v Fourierovo vrsto na intervalu $[-\pi, \pi]$ in seštej vrsti

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} \text{ in } S_2 = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$



$$\underline{a_0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi} \left[\int_{-\pi}^{-\frac{\pi}{2}} 0\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\pi} 0\,dx \right] =$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = \frac{1}{2\pi} \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \underline{\frac{1}{\pi}}$$

$$\underline{a_n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \cdot \cos(nx)\,dx = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\cos(nx - x) + \cos(nx + x) \right] dx =$$

$$\cos x \cdot \cos y = \frac{1}{2} \cos(x+y) + \cos(x-y)$$

$$\int \cos(kx)\,dx = \frac{1}{k} \sin(kx)$$

$$\underset{n>1}{=} \frac{1}{2\pi} \left[\frac{1}{n-1} \sin((n-1)x) + \frac{1}{n+1} \sin((n+1)x) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 \cdot \frac{1}{2\pi} \left[\frac{1}{n-1} \sin((n-1)x) + \frac{1}{n+1} \sin((n+1)x) \right]_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{\pi} \left[\frac{1}{n-1} \sin\left(\frac{(n-1)\pi}{2}\right) + \frac{1}{n+1} \sin\left(\frac{(n+1)\pi}{2}\right) \right] = \frac{1}{\pi} \left[\frac{1}{2m-1} \underset{(-1)^{m+1}}{\sin \frac{(2m-1)\pi}{2}} + \frac{1}{2m+1} \underset{(-1)^m}{\sin \frac{(2m+1)\pi}{2}} \right] =$$

$$a_n = \begin{cases} 0\,; & n = 2m+1 \\ \dots\,; & n = 2m \end{cases}$$

$$= \frac{1}{\pi} \left[\frac{(-1)^{m+1}}{2m-1} + \frac{(-1)^m}{2m+1} \right] =$$

$$= \frac{(2m+1)(-1)^{m+1} + (2m-1)(-1)^m}{\pi(4m^2-1)} = \frac{(-1)^m (2m-1-2m-1)}{\pi(4m^2-1)} = \underline{\frac{2(-1)^{m+1}}{\pi(4m^2-1)}}$$

$$a_1 = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} (1 + \cos 2x)\,dx = \frac{1}{\pi} \left[x + \sin 2x \cdot \frac{x^2}{2} \right]_0^{\frac{\pi}{2}} = \frac{1}{\pi} \left(\frac{\pi}{2} + 0 \right) = \frac{1}{2}$$

$\underline{b_n = 0} \Leftarrow f$ soda funkcija

$$\Rightarrow \underbrace{FV(f)(x) = \frac{1}{\pi} + \frac{\cos x}{2} + \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{\pi(4m^2-1)} \cos(2mx)} = f(x) \quad \forall x$$

$$f(\pi) = f(-\pi)$$

$$f(0) = 1 = \frac{1}{\pi} + \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{\pi(4m^2-1)}$$

$$\Rightarrow \underline{S_1} = \frac{(1 - \frac{1}{\pi} - \frac{1}{2})\pi}{-2} = \frac{(\frac{1}{2} - \frac{1}{\pi})\pi}{-2} = \frac{\frac{\pi}{2} - 1}{-2} = \underline{\frac{1}{2} - \frac{\pi}{4}}$$

$$f\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\pi} + \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{\pi(4m^2-1)} \underbrace{\cos(m\pi)}_{(-1)^m} = \frac{1}{\pi} + \sum_{m=1}^{\infty} \frac{-2}{\pi(4m^2-1)}$$

$$\Rightarrow \underline{S_2} = \frac{-\frac{1}{\pi}}{-\frac{2}{\pi}} = \underline{\frac{1}{2}}$$

4. Naj bo $f(x) = x^2$ za $x \in [0, \pi]$. Razvij funkcijo f

a) v Fourierovo vrsto na intervalu $[0, \pi]$ ter skiciraj graf vsote vrste za vse $x \in \mathbf{R}$.

b) v kosinusno Fourierovo vrsto na intervalu $[-\pi, \pi]$ ter skiciraj graf vsote vrste za vse $x \in \mathbf{R}$.

c) v sinusno Fourierovo vrsto na intervalu $[-\pi, \pi]$ ter skiciraj graf vsote vrste za vse $x \in \mathbf{R}$.

Kosinusna Fourierova vrsta je vrsta sode razširitve.

(b)

$$f_s(x) = \begin{cases} f(x), & x \geq 0 \\ f(-x); & x < 0 \end{cases}$$

$\hookrightarrow f_s : [-\pi, \pi]$

$$\underline{a_0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_s(x)\,dx = \frac{1}{\pi} \int_0^{\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{\pi} x^2\,dx = \underline{\frac{\pi^2}{3}}$$

$$\underline{a_n} = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\,dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx)\,dx = \underline{\frac{4(-1)^n}{n^2}}$$

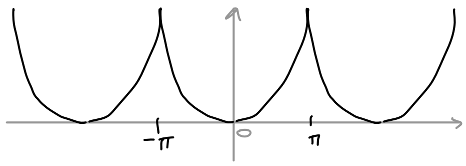
$$\int x^2 \cos(nx)\,dx = \frac{x^2 \sin(nx)}{n} - \frac{1}{n} \int x \sin(nx) = \frac{x^2}{n} \sin(nx) - \frac{2}{n}\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right) + C$$

$$= \left(\frac{x^2}{n} - \frac{2}{n^2}\right) \sin(nx) + \frac{2x}{n^2} \cos(nx) + C$$

$$\Rightarrow \underbrace{f_s(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)}$$

f_s zvezna

$f_s(\pi) = f_s(-\pi)$



(c) Sinusna vrsta je razvoj lihe razširitve.

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx = \frac{2}{\pi} \int_0^{\pi} x^2 \sin(nx)\,dx = \frac{2}{\pi}\left(-\frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right)$$

$$\int x^2 \sin(nx)\,dx \underset{\uparrow}{=} -\frac{x^2}{n}\cos(nx) + \frac{2}{n} \int x \cos(nx)\,dx = -\frac{x^2}{n}\cos(nx) + \frac{2}{n}\left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right) + C$$

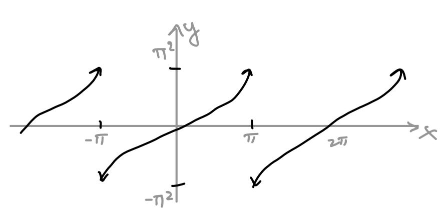
PER PARTES

$u = x^2 \quad dv = \sin(nx)dx$

$$= \left(\frac{2}{n^3} - \frac{x^2}{n}\right)\cos(nx) + \frac{2x}{n^2}\sin(nx) + C$$

$$\Rightarrow \underbrace{f_L(x) = \sum_{n=1}^{\infty} b_n \sin(nx)} \quad ; \quad x \in (-\pi, \pi)$$

sicer: vsota $= \dfrac{f_L(\pi) + f_L(-\pi)}{2} = 0$



5. Razvij $f(x) = x(\pi - x)$ v sinusno Fourierovo vrsto na intervalu $[-\pi, \pi]$ in seštej vrsto $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} = S$

Iz prejšnje naloge vemo: $\underset{x\in[0,\pi)}{x^2} = \sum_{n=1}^{\infty} \left(\frac{-2\pi(-1)^n}{n} + \frac{4}{\pi}\frac{(-1)^n-1}{n^3} \right) \sin(nx)$

$$\Rightarrow FV_{\sin}(-x^2) = \sum_{n=1}^{\infty} \left(\frac{2\pi(-1)^n}{n} + \frac{4(1-(-1)^n)}{n^3\pi} \right) \sin(nx)$$

$FV_{\sin}(\pi x)$:

$$b_n = \frac{1}{\pi}\int_0^{\pi} \pi x \sin(nx)\,dx = 2\left(-\frac{\pi}{n}(-1)^n\right)$$

$$\int x \sin(nx)\,dx = -\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx) + C$$

$$\Rightarrow FV_{\sin}(f)(x) = \sum_{n=1}^{\infty} \left(\frac{2\pi(-1)^n}{n} + \frac{4(1-(-1)^n)}{n^3\pi} - \frac{2\pi}{n}(-1)^n \right) \sin(nx) =$$

$$= \sum_{n=1}^{\infty} \underbrace{\frac{4(1-(-1)^n)}{n^3\pi}}_{=0 \text{ za sode } n} \sin(nx) =$$

$$= \sum_{m=1}^{\infty} \frac{8}{\pi(2m-1)^3} \sin((2m-1)x)$$

$$x = \frac{\pi}{2}: \quad f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} = \sum_{m=1}^{\infty} \frac{8}{\pi(2m-1)^3} \sin\left((2m-1)\cdot\frac{\pi}{2}\right) =$$

$$= \sum_{m=1}^{\infty} \frac{8}{\pi} \frac{(-1)^{m+1}}{(2m-1)^3}$$

$$\Rightarrow S = \frac{\pi^2}{4} \cdot \frac{\pi}{-8} = -\frac{\pi^3}{32}$$

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6. a) Razvij $f(x) = \cos x$ v Fourierovo vrsto na intervalu $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ter skiciraj graf vsote vrste za vse $x \in \mathbf{R}$.

b) Razvij funkcijo $f(x) = \sin^3 x$ v Fourierovo vrsto na intervalu $[-\pi, \pi]$.

(b) $f(x) = \sin^3 x$ liha funkcija $\Rightarrow a_n = 0 \quad \forall n \in \mathbb{N}_0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3 x \cdot \sin(nx)\,dx$$

$\sin^2 x = \underbrace{\frac{1}{2}(1-\cos(2x))}_{\text{uporabimo v } FV(\sin^3)(x)}$

$$\sin^3 x = \sin x \cdot \sin^2 x = \sin x \cdot \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) = \frac{1}{2}\sin x - \frac{1}{2}\sin x\cos(2x) =$$

$\sin x \cdot \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

$$= \frac{1}{2}\sin x - \frac{1}{4}\sin(3x) - \frac{1}{4}\sin x =$$

$$FV(\sin^3)(x) = -\frac{1}{4}\sin(3x) + \frac{1}{4}\sin x$$

$$\Rightarrow b_1 = \frac{1}{4}, \quad b_3 = -\frac{1}{4}, \quad \text{ostali } b_n = 0$$

Opomba:

Na ta način bi lahko razvili $\underset{\text{POLINOM}}{P}(\sin x, \cos x)$.

7. Prepričaj se, da za vsak $x \in [-\pi, \pi]$ velja

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

in nato s pomočjo Parsevalove enakosti seštej vrsto

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Poznamo kosinusno Fourierovo vrsto za x^2: $FV_{cos}(x^2) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$

$$FV(f)(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

Parseval: $\int_{-\pi}^{\pi} f(x)^2 dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$

$$\int_{-\pi}^{\pi} x^4 dx = \frac{x^5}{5}\Big|_{-\pi}^{\pi} = 2\frac{\pi^5}{5} = 2\pi\left(\frac{\pi^2}{3}\right)^2 + \underbrace{\pi \sum_{n=1}^{\infty} \frac{16}{n^4}}_{16\pi \cdot S}$$

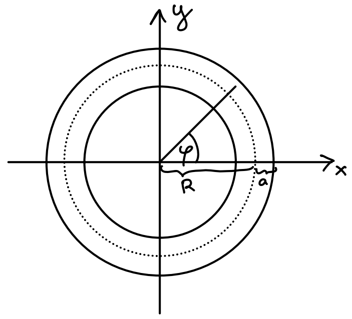
$$\Rightarrow S = \frac{\pi^4}{90}$$

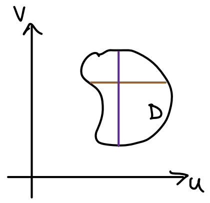
KRIVULJE IN PLOSKVE V PROSTORU

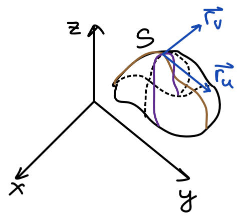
10. Izračunaj površino torusa s polmeroma $0 < a < R$.

površina ploskve: $S = \int_D \underbrace{\sqrt{EG - F^2}}_{|\vec{r}_u \times \vec{r}_v|} \, du \, dv$

$E = \vec{r}_u \cdot \vec{r}_u$
$F = \vec{r}_u \cdot \vec{r}_v$
$G = \vec{r}_v \cdot \vec{r}_v$







torusne koordinate: $x = (R + r\cos\vartheta)\cos\varphi$
$y = (R + r\cos\vartheta)\sin\varphi$
$z = r\sin\vartheta$

$\Rightarrow$ parametrizacija površine torusa $(r = a)$:

$\vec{r}(\varphi, \vartheta) = \big((R + a\cos\vartheta)\cos\varphi, \ (R + a\cos\vartheta)\sin\varphi, \ a\sin\vartheta\big) \quad \varphi, \vartheta \in [0, 2\pi)$

$\vec{r}_\varphi(\varphi, \vartheta) = \big(-(R + a\cos\vartheta)\sin\varphi, \ (R + a\cos\vartheta)\cos\varphi, \ 0\big)$

$\vec{r}_\vartheta(\varphi, \vartheta) = \big(-a\sin\vartheta\cos\varphi, \ -a\sin\vartheta\sin\varphi, \ a\cos\vartheta\big)$

$\Rightarrow E = (R + a\cos\vartheta)^2$
$F = 0$
$G = a^2$

$\vec{r}_u \perp \vec{r}_v \Leftrightarrow$ koordinatni krivulji sta pravokotni

$$\underline{P(S)} = \int_0^{2\pi} d\varphi \int_0^{2\pi} \sqrt{a^2 (R + a\cos\vartheta)^2} \, d\vartheta = 2\pi \int_0^{2\pi} a(R + a\cos\vartheta) \, d\vartheta = 2\pi \cdot 2\pi \cdot aR = 4\pi^2 aR = \underline{= (2\pi R)(2\pi a)}$$

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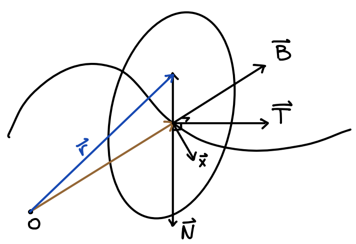
7. Naj bo K gladka krivulja z dolžino l in $a > 0$ dovolj majhen. Definirajmo telo D kot unijo vseh normalnih krogov na krivuljo K s polmerom a (D je 'odebeljena' krivulja K). Dokaži, da je tedaj prostornna telesa D enaka $\pi a^2 \cdot l$. Izračunaj površino plašča telesa D.

K parametriziramo z naravnim parametrom:

$\vec{r} = \vec{r}_K(s) \ ; \ s \in [0, l]$



$(\vec{T}, \vec{N}, \vec{B})$ ... spremljajoči trieder

$\vec{x} = r(\cos\varphi \vec{N} + \sin\varphi \vec{B}) \ ; \ \varphi \in [0, 2\pi], \ r \in [0, a]$

$\underline{\vec{r}(s, r, \varphi)} = \vec{r}_K(s) + \vec{x} =$
$= \underline{\vec{r}_K(s) + r\cos\varphi \vec{N}(s) + r\sin\varphi \vec{B}(s)}$

$s \in [0, l], \ r \in [0, a], \ \varphi \in [0, 2\pi]$

$$D\vec{r} = \frac{\partial \vec{r}}{\partial(s,r,\varphi)} = \begin{array}{c} \begin{bmatrix} x_s & x_r & x_\varphi \\ y_s & y_r & y_\varphi \\ z_s & z_r & z_\varphi \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \vec{r}_s & \vec{r}_r & \vec{r}_\varphi \end{matrix} \end{array}$$

mešani produkt

$$|\det D\vec{r}| = |\det(\vec{r}_s, \vec{r}_r, \vec{r}_\varphi)| = |[\vec{r}_s, \vec{r}_r, \vec{r}_\varphi]| = |(\vec{r}_s \times \vec{r}_\varphi)\cdot \vec{r}_r|$$

$$\vec{r}_s = \underbrace{\vec{r}_K{}'(s)}_{\vec{T}} + r\cos\varphi\, \vec{N}'(s) + r\sin\varphi\, \vec{B}'(s)$$

$$\vec{r}_r = \cos\varphi\, \vec{N}(s) + \sin\varphi\, \vec{B}(s)$$

$$\vec{r}_\varphi = -r\sin\varphi\, \vec{N}(s) + r\cos\varphi\, \vec{B}(s)$$

$$\begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}' = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & \omega \\ 0 & -\omega & 0 \end{bmatrix} \cdot \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

pozitivno orientirana ortonormirana baza

$$\vec{N}' = -K\vec{T} + \omega\vec{B}$$
$$\vec{B}' = -\omega\vec{N}$$

$\vec{T} \to \vec{N} \to \vec{B} \to \vec{T}$

$$\vec{r}_s \times \vec{r}_\varphi = \left[(1 - Kr\cos\varphi)\vec{T} + (-\omega r\sin\varphi)\vec{N} + (\omega r\cos\varphi)\vec{B}\right] \times \left[-r\sin\varphi\,\vec{N} + r\cos\varphi\,\vec{B}\right] =$$

$$= \left[\underbrace{(-\omega r^2\sin\varphi\cos\varphi + \omega r^2\sin\varphi\cos\varphi)}_{0}\vec{T} + r\cos\varphi(Kr\cos\varphi - 1)\,\vec{N} + (Kr\cos\varphi - 1)r\sin\varphi\,\vec{B}\right] =$$

$$= r\,(Kr\cos\varphi - 1)(\cos\varphi\,\vec{N} + \sin\varphi\,\vec{B})$$

$$\underline{(\vec{r}_s \times \vec{r}_\varphi)\cdot\vec{r}_r} = \left[r\,(Kr\cos\varphi - 1)(\cos\varphi\,\vec{N} + \sin\varphi\,\vec{B})\right]\cdot\left[\cos\varphi\,\vec{N} + \sin\varphi\,\vec{B}\right] = \underline{r(Kr\cos\varphi - 1)}$$

$$V = \int_0^\ell ds \int_0^a dr \int_0^{2\pi} \det D\vec{r}\, d\varphi = \int_0^\ell ds \int_0^a dr \int_0^{2\pi} r\,\underbrace{(1 - \overbrace{Kr\cos\varphi}^{=0})}_{\leq 1 \ldots \text{"}|\ldots|\text{"}}\, d\varphi =$$

, če $Kr\cos\varphi < 1 \Leftarrow Kr < 1 \Leftrightarrow r < \frac{1}{K}$

polmer pritisnjene krožnice $\downarrow$

$$= \int_0^\ell ds \int_0^a r\,dr \int_0^{2\pi} d\varphi = 2\pi\,\ell\,\frac{a^2}{2} = \underline{(\pi a^2)\,\ell}$$

parametrizacija plašča = parametrizacija pri $r = a$:

$$\vec{r}(s,\varphi) = \vec{r}_K(s) + a\cos\varphi\,\vec{N} + a\sin\varphi\,\vec{B}$$

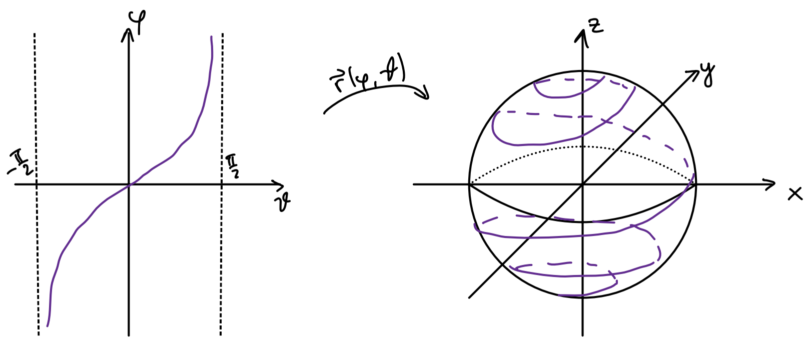
$$\vec{r}_s \times \vec{r}_\varphi = a(Ka\cos\varphi - 1)(\cos\varphi\,\vec{N} + \sin\varphi\,\vec{B})$$

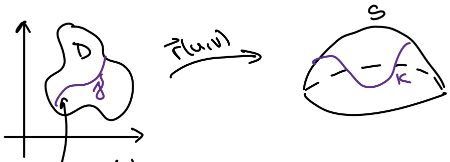
$$|\vec{r}_s \times \vec{r}_\varphi| = a(1 - Ka\cos\varphi)$$

$$\underline{P(S)} = \int_D |\vec{r}_s \times \vec{r}_\varphi|\, ds\, d\varphi = \int_0^\ell ds \int_0^{2\pi} a(1 - \underbrace{Ka\cos\varphi}_{\text{"}0\text{"}})\, d\varphi = \underline{(2\pi a)\,\ell}$$

---

11. Na enotski sferi s središčem v $(0,0,0)$, ki jo parametriziramo s sferičnima koordinatama φ in ϑ, leži krivulja K, podana z zvezo $\varphi = \operatorname{tg}\vartheta$. Določi dolžino krivulje K.





$u = u(t)$
$v = v(t)$

$$ds^2 = E\,du^2 + G\,dv^2 + 2F\,du\,dv \qquad du = \dot{u}\,dt, \quad dv = \dot{v}\,dt$$

$$\Rightarrow \ell(K) = \int_{t_1}^{t_2} ds$$

$$\varphi = \tan\vartheta$$
$$\vartheta = \vartheta$$

parametrizacija sfere:

$$\vec{r}(\varphi, \vartheta) = (\cos\vartheta\cos\varphi,\ \cos\vartheta\sin\varphi,\ \sin\vartheta)$$

$$\vec{r}_\varphi(\varphi, \vartheta) = (-\cos\vartheta\sin\varphi,\ \cos\vartheta\cos\varphi,\ 0)$$
$$\vec{r}_\vartheta(\varphi, \vartheta) = (-\sin\vartheta\cos\varphi,\ -\sin\vartheta\cos\varphi,\ \cos\vartheta)$$

$E = \vec{r}_\varphi \vec{r}_\varphi = \cos^2\vartheta\sin^2\varphi + \cos^2\vartheta\cos^2\varphi = \cos^2\vartheta$

$F = 0$ ("poldnevniki in vzporedniki so pravokotni - usmeritvi $\vec{r}_\varphi$ in $\vec{r}_\vartheta$)

$G = \vec{r}_\vartheta \vec{r}_\vartheta = \sin^2\vartheta\cos^2\varphi + \sin^2\vartheta\sin^2\varphi + \cos^2\vartheta = 1$

$$ds^2 = \cos^2\vartheta\,d\varphi^2 + d\vartheta^2 = \frac{1}{\cos^2\vartheta}\,d\vartheta^2 + d\vartheta^2$$

$$d\varphi = \frac{d}{d\vartheta}\varphi\,d\vartheta = \frac{1}{\cos^2\vartheta}\,d\vartheta$$

$$\ell(K) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{1+\cos^2\vartheta}{\cos^2\vartheta}}\,d\vartheta = 2\int_0^{\frac{\pi}{2}} \frac{\sqrt{1+\cos^2\vartheta}}{\cos\vartheta}\,d\vartheta$$

POSPLOŠENI INTEGRAL

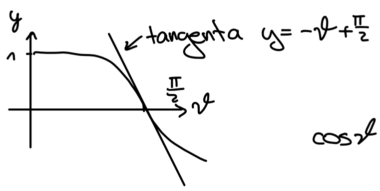
Ali integral konvergira?

$$f(x) = \frac{g(x)}{(x-a)^s}$$

1) $s < 1 \wedge \lim\limits_{x\to a} g(x) \in \mathbb{R} \Rightarrow \int_a^{\dots} f(x)\,dx$ obstaja pri a

2) $s \geq 1 \wedge \lim\limits_{x\to a} g(x) \neq 0 \Rightarrow \int_a^{\dots} f(x)\,dx$ ne obstaja

$$\frac{\sqrt{1+\cos^2\vartheta}}{\cos\vartheta} = \frac{g(x)}{(\frac{\pi}{2}-\vartheta)^s}$$



$\cos\vartheta \approx \frac{\pi}{2} - \vartheta$ blizu $\frac{\pi}{2}$

$s = 1$

$$f(\vartheta) = \frac{\sqrt{1+\cos^2\vartheta}\,(\frac{\pi}{2}-\vartheta)}{\cos\vartheta\,(\frac{\pi}{2}-\vartheta)}, \qquad \frac{\sqrt{1+\cos^2\vartheta}\,(\frac{\pi}{2}-\vartheta)}{\cos\vartheta} = g(\vartheta)$$

L'HOPITAL

$$\lim_{\vartheta\nearrow\frac{\pi}{2}} \frac{\overset{\nearrow 1}{\sqrt{1+\cos^2\vartheta}}\,(\frac{\pi}{2}-\vartheta)}{\cos\vartheta} = \lim_{\vartheta\nearrow\frac{\pi}{2}} \frac{-1}{-\sin\vartheta} = 1 \neq 0$$

$$\lim g(\vartheta) \neq 0 \wedge s = 1 \Rightarrow \underbrace{\ell(K) = \infty}$$

Krivulja je neskončne dolžine.

9. Naj bo f zvezno odvedljiva funkcija in S ploskev podana z zvezo

$$z = xf(x/y),$$

kjer je $x, y > 0$. Dokaži, da imajo vse tangentne ravnine na ploskev S skupno točko.

Normala ravnine Π: $\vec{n} = \vec{r}_u \times \vec{r}_v$

Naš primer: $\vec{r}(x,y) = (x, y, \underbrace{x f(\tfrac{x}{y})}_{z(x,y)})$

$$\vec{r}_x = (1, 0, z_x)$$
$$\vec{r}_y = (0, 1, z_y) \qquad \Rightarrow \vec{n} = (-z_x, -z_y, 1)$$

Opomba: $z = z(x,y) \Rightarrow \vec{n} = (z_x, z_y, -1)$

$$z_x = f(\tfrac{x}{y}) + \tfrac{x}{y} f'(\tfrac{x}{y}) \qquad z_y = x f'(\tfrac{x}{y})(-\tfrac{x}{y^2})$$

$$\Rightarrow \vec{n} = \left(f(\tfrac{x}{y}) + \tfrac{x}{y} f'(\tfrac{x}{y}),\ -\tfrac{x^2}{y^2} f'(\tfrac{x}{y}),\ -1 \right)$$

$T_0(x_0, y_0, \underbrace{x_0 f(\tfrac{x_0}{y_0})}_{z_0})$

Enačba ravnine Π: $(x, y, z)\vec{n} = (x_0, y_0, z_0)\vec{n}$

$$\left(f(\tfrac{x_0}{y_0}) + \tfrac{x_0}{y_0} f'(\tfrac{x_0}{y_0})\right)x + \left(-\tfrac{x_0^2}{y_0^2} f'(\tfrac{x_0}{y_0})\right)y - z = \left(f(\tfrac{x_0}{y_0}) + \tfrac{x_0}{y_0} f'(\tfrac{x_0}{y_0})\right)x_0 - \tfrac{x_0^2}{y_0} f'(\tfrac{x_0}{y_0}) - x_0 f(\tfrac{x_0}{y_0}) = 0$$

$\Rightarrow$ vse tangentne ravnine vsebujejo točko O

KRIVULJNI IN PLOSKOVNI INTEGRALI

1. Naj bo $\vec{a} \in \mathbf{R}^3$ izbran vektor.
   a) Dokaži, da je gradient polja $\vec{f}(\vec{r}) = \frac{1}{|\vec{r}-\vec{a}|}$ solenoidalno polje.
   b) Izračinaj rotor polja $\vec{f}(\vec{r}) = \vec{r} \times \vec{a}$.

$f(\vec{r})$ ... skalarno polje ← $u: \mathbb{R}^3 \to \mathbb{R}$

$$\operatorname{grad} u = \vec{\nabla} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \quad ; \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

a) $\vec{f}(\vec{r})$ je solenoidalno polje $\Leftrightarrow$ $\operatorname{div} \vec{f}(\vec{r}) = 0$

$$\vec{f} = (P, Q, R) \Rightarrow \vec{\nabla}\vec{f} = P_x + Q_y + R_z = \operatorname{div}\vec{f}$$

$\vec{r} = (x,y,z)$, $\vec{a} = (a,b,c)$:

$$f(x,y,z) = \frac{\wedge}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

$$\operatorname{grad} f = \Big[-\tfrac{1}{2}\cdot 2(x-a)\big((x-a)^2+(y-b)^2+(z-c)^2\big)^{-\frac{3}{2}}, -\tfrac{1}{2}\cdot 2(y-b)\big((x-a)^2+(y-b)^2+(z-c)^2\big)^{-\frac{3}{2}},$$
$$-\tfrac{1}{2}\cdot 2(z-c)\big((x-a)^2+(y-b)^2+(z-c)^2\big)^{-\frac{3}{2}}\Big]$$

$$\operatorname{grad} f = |\vec{r}-\vec{a}|^{-3}(a-x, b-y, c-z) = |\vec{r}-\vec{a}|^{-3}(\vec{a}-\vec{r})$$

$\underline{\operatorname{div}(\operatorname{grad} f) = 0}$

$$\operatorname{div}(\operatorname{grad} f) = X_x + Y_y + Z_z =$$
$$= \left[-|\vec{r}-\vec{a}|^{-3} - (x-a)\cdot 2(x-a)\left(-\tfrac{3}{2}\right)|\vec{r}-\vec{a}|^{-5}\right] + \left[-|\vec{r}-\vec{a}|^{-3} - (y-b)\cdot 2(y-b)\left(-\tfrac{3}{2}\right)|\vec{r}-\vec{a}|^{-5}\right] +$$
$$+ \left[-|\vec{r}-\vec{a}|^{-3} - (z-c)\cdot 2(z-c)\left(-\tfrac{3}{2}\right)|\vec{r}-\vec{a}|^{-5}\right] =$$
$$= -3|\vec{r}-\vec{a}|^{-3} + 3|\vec{r}-\vec{a}|^{-5}\big(\underbrace{(x-a)^2+(y-b)^2+(z-c)^2}_{|\vec{r}-\vec{a}|^2}\big) =$$
$$= 0$$

$\Rightarrow$ $\underline{\operatorname{grad} \vec{f}(\vec{r}) \text{ je solenoidalno polje}}$

b) $\vec{f}(\vec{r}) = \vec{r} \times \vec{a}$; $\vec{r} = (x,y,z)$, $\vec{a} = (a,b,c)$

$$\operatorname{rot} \vec{f}(\vec{r}) = \vec{\nabla} \times \vec{f}(\vec{r})$$

$$\vec{f}(x,y,z) = (yc - bz, az - cx, xb - ay)$$

$$\operatorname{rot} \vec{f}(x,y,z) = (-a-a, -b-b, -c-c) = -2\vec{a}$$

$\Rightarrow$ $\underline{\operatorname{rot}(\vec{r}\times\vec{a}) = -2\vec{a}}$

OPOZORILO: $\operatorname{rot}(\vec{r}\times\vec{a}) = \vec{\nabla}\times(\vec{r}\times\vec{a}) = \underbrace{(\vec{\nabla}\vec{a})}_{0}\vec{r} - \underbrace{(\vec{\nabla}\vec{r})}_{3}\vec{a} = \cancel{-3\vec{a}}$ ← NAPAČNO!!!

2. Določi vse take zvezno odvedljive funkcije $f : \mathbf{R} \to \mathbf{R}$, da bo

$$\vec{f}(x,y,z) = ((1+x^2)f(x), -2xyf(x), -3z)$$

solenoidalno polje.

$\operatorname{div} \vec{f}(x,y,z) = 0$ ← to mora veljati

$\operatorname{div} \vec{f}(x,y,z) = 2x f(x) + (1+x^2) f'(x) - 2x f(x) - 3 = 0$

$$f'(x) = \frac{3}{1+x^2}$$

$$\int f'(x)\,dx = 3\arctan x + C = f(x)$$

---

3. a) Dokaži, da je $\vec{f}(x,y,z) = (2x\cos y - y^2 \sin x, 2y\cos x - x^2 \sin y, 4)$ potencialno in določi njegov potencial.
   b) Določi konstanti a in b tako, da bo polje

$$\vec{f}(x,y,z) = (2(axzy^4 - y), 2(bx^2zy^3 - x), 3x^2y^4)$$

potencialno in določi njegov potencial.

$\vec{f} : \mathbb{R}^3 \to \mathbb{R}^3$ ... vektorsko polje

$\vec{f}$ je potencialno, če ima potencial $u : \mathbb{R}^3 \to \mathbb{R}$, t.j. $\vec{f} = \operatorname{grad} u = \vec{\nabla} u = (u_x, u_y, u_z)$

(a) Poiščimo potencial u:

$u_x = 2x\cos y - y^2 \sin x$ (1)
$u_y = 2y\cos x - x^2 \sin y$ (2)
$u_z = 4$ (3)

(1) $\Rightarrow u = x^2\cos y + y^2\cos x + C_1(y,z)$
integriramo po x

integriramo po y
(2) $\Rightarrow u = y^2\cos x + x^2\cos y + C_2(x,z)$

integriramo po z
(3) $\Rightarrow u = 4z + C_3(x,y)$

vzamemo "unijo" členov

$\Rightarrow u = x^2\cos y + y^2\cos x + 4z + C \quad ; C \in \mathbb{R}$

potencial $\vec{f}$

$\Rightarrow \vec{f}$ je potencialno

Polje $\vec{f} : D \to \mathbb{R}^3$ (zvezno) je potencialno $\Leftrightarrow \operatorname{rot} \vec{f} = 0$.
zvezdasto območje

$\operatorname{rot} \vec{f} = \vec{\nabla} \times \vec{f}$

(b) $\operatorname{rot} \vec{f} = (12x^2y^3 - 2bx^2y^3,\ 2axy^4 - 6xy^4,\ 4bxzy^3 - 2 - 8axzy^3 + 2) = 0$

$12 = 2b$ $\quad$ $2a = 6$ $\quad$ $b = 2a$ ✓

$b = 6$ $\quad$ $a = 3$

Poiščimo potencial u:

$$\begin{aligned} u_x &= 2(3xzy^4 - y) &&\Rightarrow u = 3x^2zy^4 - 2yx + C_1(y,z)\\ u_y &= 2(6x^2zy^3 - x) &&\Rightarrow u = 3x^2zy^4 - 2xy + C_2(x,z)\\ u_z &= 3x^2y^4 &&\Rightarrow u = 3x^2y^4z + C_3(x,y)\end{aligned}$$

$$\Rightarrow \underline{u = 3x^2y^4z - 2xy + C}; \quad C\in\mathbb{R}$$

---

4. Naj bosta $\vec{a}$ in $\vec{b}$ konstantna vektorja. Dokaži enakost

$$\operatorname{grad}\frac{\vec{a}\vec{r}}{\vec{b}\vec{r}} = \frac{\vec{r}\times(\vec{a}\times\vec{b})}{(\vec{b}\vec{r})^2}.$$

$\vec{r} = (x,y,z)$; $\vec{a} = (a_1,a_2,a_3)$, $\vec{b} = (b_1,b_2,b_3)$

$$u(\vec{r}) = \frac{\vec{a}\vec{r}}{\vec{b}\vec{r}}$$

$$\Rightarrow u(x,y,z) = \frac{(a_1,a_2,a_3)\cdot(x,y,z)}{(b_1,b_2,b_3)\cdot(x,y,z)} = \frac{a_1x + a_2y + a_3z}{b_1x + b_2y + b_3z}$$

$$\Rightarrow u_x = \frac{a_1(b_1x + b_2y + b_3z) - b_1(a_1x + a_2y + a_3z)}{(b_1x + b_2y + b_3z)^2} = \frac{a_1(\vec{b}\vec{r}) - b_1(\vec{a}\vec{r})}{(\vec{b}\vec{r})^2}$$

Iz simetrije sklepamo: $u_y = \dfrac{a_2(\vec{b}\vec{r}) - b_2(\vec{a}\vec{r})}{(\vec{b}\vec{r})^2}$

$$u_z = \frac{a_3(\vec{b}\vec{r}) - b_3(\vec{a}\vec{r})}{(\vec{b}\vec{r})^2}$$

$$\underline{\operatorname{grad} u} = (u_x, u_y, u_z) = \frac{1}{(\vec{b}\vec{r})^2}\left(a_1(\vec{b}\vec{r}) - b_1(\vec{a}\vec{r}),\ a_2(\vec{b}\vec{r}) - b_2(\vec{a}\vec{r}),\ a_3(\vec{b}\vec{r}) - b_3(\vec{a}\vec{r})\right) =$$

$$= \frac{1}{(\vec{b}\vec{r})^2}\left((\vec{b}\vec{r})(a_1,a_2,a_3) - (\vec{a}\vec{r})(b_1,b_2,b_3)\right) =$$

$$= \frac{(\vec{b}\vec{r})\vec{a} - (\vec{a}\vec{r})\vec{b}}{(\vec{b}\vec{r})^2} =$$

$$= \underline{\frac{\vec{r}\times(\vec{a}\times\vec{b})}{(\vec{b}\vec{r})^2}}$$

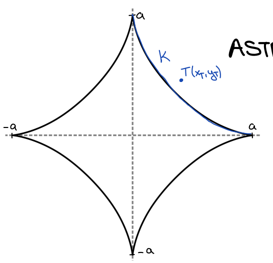
dvojni vektorski produkt: $\vec{r}\times(\vec{a}\times\vec{b}) = (\vec{r}\vec{b})\vec{a} - (\vec{r}\vec{a})\vec{b}$

---

8. Naj bo $a > 0$. Izračunaj težišče homogenega loka astroide

$$x^{2/3} + y^{2/3} = a^{2/3},$$

$x, y \geq 0$.



ASTROIDA

Krivuljo $K\subseteq\mathbb{R}^3$ oz. $K\subseteq\mathbb{R}^2$ parametriziramo:
$\vec{r} = \vec{r}(t)$; $t\in[t_1,t_2]$
$u: \mathbb{R}^3\to\mathbb{R}$, definirano na K

$$\int_K u\,dS = \int_{t_1}^{t_2} u(\vec{r}(t))\,\underbrace{|\dot{\vec{r}}(t)|\,dt}_{ds}$$

Težišče $(K\subseteq\mathbb{R}^3)$: $x_T = \dfrac{\int_K x\,dm}{\underbrace{\int_K dm}_{=m(K)}}$ ← diferencial mase

$dm = \rho\cdot ds$, $\rho = \rho(x,y,z)$ ... dolžinska gostota

K homogena $\Leftrightarrow \rho = \rho_0$

$$\Rightarrow \underline{x_T = \frac{\int_K x\rho(x,y,z)\,ds}{\int_K \rho\,ds}}$$

$$\Rightarrow \underline{x_T} = \frac{\int_K x\,ds}{\underbrace{\int_K ds}_{=\ell}} = \underline{\frac{1}{\ell}\int_K x\,ds}$$

Parametrizacija astroide:

$$x = a\cdot\cos^3 t \qquad ; \; t\in\left[0,\tfrac{\pi}{2}\right] \qquad \dot{x} = -3a\cos^2 t\sin t$$
$$y = a\cdot\sin^3 t \qquad\qquad\qquad\qquad \dot{y} = 3a\sin^2 t\cos t$$
$$\Rightarrow ds = |\dot{\vec{r}}(t)|dt = 3a\cos t\sin t\, dt$$

$$\underline{\ell(K)} = \int_K ds = \int_0^{\frac{\pi}{2}} 3a\cos t\sin t\, dt = \frac{3}{2}a\int_0^{\frac{\pi}{2}}\sin(2t)\,dt = \frac{3}{4}a\left(-\cos(2t)\right)\Big|_0^{\frac{\pi}{2}} = \underline{\frac{3}{2}a}$$

$$\underline{\int_K x\,ds} = \int_0^{\frac{\pi}{2}} 3a^2\cos^4 t\sin t\, dt = 3a^2\cdot\frac{1}{2}B\left(\tfrac{5}{2},1\right) = \frac{3}{2}a^2\frac{\Gamma(\frac{5}{2})\Gamma(1)}{\Gamma(\frac{7}{2})} = \frac{3}{2}a^2\frac{2}{5} = \underline{\frac{3}{5}a^2}$$

$$2p-1=4$$
$$2q-1=1$$

$$\Rightarrow \underline{x_T} = \frac{\frac{3}{5}a^2}{\frac{3}{2}a} = \underline{\frac{2}{5}a}$$

Simetrično: $\underline{y_T = \frac{2}{5}a}$

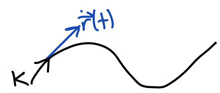
---

9. Dani sta števili $a>0$ in α ter krivulja $K = S(0,a)\cap\Pi$, kjer je $S(a,0)$ sfera s središčem v $(0,0,0)$ in polmerom a ter Π ravnina z enačbo $y = x\,\mathrm{tg}\alpha$. Izračunaj integral

$$I = \int_K \underbrace{(y-z)}_{P}dx + \underbrace{(z-x)}_{Q}dy + \underbrace{(x-y)}_{R}dz$$

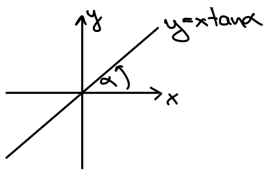
a) direktno.

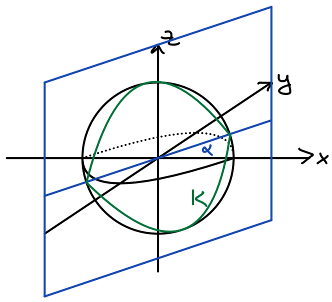
(a) $(dx, dy, dz) = d\vec{r}$ $\qquad \vec{r} = \vec{r}(t),\; t\in[t_1,t_2]$ parametrizacija K ustreza izbrani orientaciji:



$$I = \int_K \underbrace{(P,Q,R)}_{\vec{f}}\,d\vec{r} = \int_{t_1}^{t_2}\vec{f}(\vec{r}(t))\cdot\dot{\vec{r}}(t)\,dt$$

Π: $y = x\cdot\tan\alpha$





$$S(0,a):\quad x = a\cos\varphi\cos\vartheta \qquad \vartheta\in\left[-\tfrac{\pi}{2},\tfrac{\pi}{2}\right]$$
$$y = a\sin\varphi\cos\vartheta \qquad \varphi = \alpha \leftarrow \text{opazimo}$$
$$z = a\sin\vartheta$$

$$\vec{r}(\vartheta) = (a\cos\alpha\cos\vartheta,\; a\sin\alpha\cos\vartheta,\; a\sin\vartheta) \qquad \text{za } \vartheta\in[0,2\pi] \text{ opišemo celotno } K.$$

$$\dot{\vec{r}}(\vartheta) = (-a\cos\alpha\sin\vartheta, -a\sin\alpha\sin\vartheta,\; a\cos\vartheta)$$

$$\vec{f}(\vec{r}(\vartheta))\cdot\dot{\vec{r}}(\vartheta) = (a\sin\alpha\cos\vartheta - a\sin\vartheta,\; a\sin\vartheta - a\cos\alpha\cos\vartheta,\; a\cos\alpha\cos\vartheta - a\cos\vartheta\sin\alpha)\cdot$$
$$\cdot(-a\cos\alpha\sin\vartheta, -a\sin\alpha\sin\vartheta,\; a\cos\vartheta) =$$

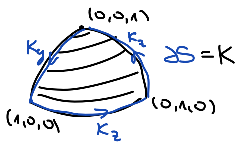
$$= a^2(\cos\alpha\sin^2\vartheta - \sin\alpha\cos\alpha\sin\vartheta\cos\vartheta) + a^2(\sin\alpha\cos\alpha\sin\vartheta\cos\vartheta - \sin\alpha\sin^2\vartheta) +$$
$$+ a^2(\cos\alpha\cos^2\vartheta - \sin\alpha\cos^2\vartheta)$$

$$= a^2(\cos\alpha - \sin\alpha)$$

$$\underline{\int_K \vec{f}\,d\vec{r}} = \int_0^{2\pi} a^2(\cos\alpha - \sin\alpha)\,d\vartheta = \underline{2\pi a^2(\cos\alpha - \sin\alpha)}$$

10. Naj bo $\vec{c} = (p, q, r)$ in $\vec{f}(\vec{r}) = \vec{c} \times \vec{r}$. Izračunaj cirkulacijo polja $\vec{f}$ vzdolž roba ploskve $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1,\ x, y, z \geq 0\}$.

S ... del sfere v I. oktantu prostora



$\partial S = K$ (odsekoma gladka krivulja: $K = \underbrace{K_x \cup K_y \cup K_z}_{\text{gladke grivulje}}$)

CIRKULACIJA POLJA $\vec{f}$: $\int_K \vec{f}\, d\vec{r}$

$$K_z:\ \left.\begin{aligned} x &= \cos\varphi \\ y &= \sin\varphi \\ z &= 0 \end{aligned}\right\} \Rightarrow \vec{r}(\varphi) = (\cos\varphi, \sin\varphi, 0),\quad \varphi \in [0, \tfrac{\pi}{2}]$$

prava orientacija

$$\dot{\vec{r}}(\varphi) = (-\sin\varphi, \cos\varphi, 0)$$

$$\vec{f}(x,y,z) = (qz - ry,\ rx - pz,\ py - qx) = (-r\sin\varphi,\ r\cos\varphi,\ p\sin\varphi - q\cos\varphi)$$

$$\vec{f}(\vec{r}(\varphi))\cdot\dot{\vec{r}}(\varphi) = r\sin^2\varphi + r\cos^2\varphi = r$$

$$\Rightarrow \int_{K_z} \vec{f}\, d\vec{r} = \int_0^{\frac{\pi}{2}} r\, d\varphi = \frac{\pi}{2} r$$

Simetrično dobimo: $\int_{K_x} \vec{f}\, d\vec{r} = \frac{\pi}{2} p$ in $\int_{K_y} \vec{f}\, d\vec{r} = \frac{\pi}{2} q$

$$\Rightarrow \int_K \vec{f}\, d\vec{r} = \int_{K_z} \vec{f}\, d\vec{r} + \int_{K_x} \vec{f}\, d\vec{r} + \int_{K_y} \vec{f}\, d\vec{r} = \frac{\pi}{2}(p+q+r)$$

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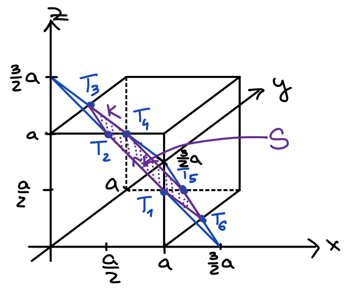
7. Naj bo $a > 0$ in $\vec{f}(x, y, z) = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$. Izračunaj cirkulacijo polja $\vec{f}$ vzdolž preseka roba kocke $[0, a]^3$ in ravnine z enačbo $x + y + z = \frac{3a}{2}$.



Računamo: $\oint_K \vec{f}\, d\vec{r}$

Parametrizacija daljice (med A in B): $\vec{r}(t) = (1-t)\vec{a} + t\vec{b}$, $t \in [0,1]$

$$\vec{r}_1(t) = (1-t)(a, 0, \tfrac{a}{2}) + t(\tfrac{a}{2}, 0, a) = a\left(\frac{2-t}{2}, 0, \frac{1+t}{2}\right) \longrightarrow \dot{\vec{r}}_1(t) = a\left(-\tfrac{1}{2}, 0, \tfrac{1}{2}\right)$$

$$I_1 = \int_{T_1T_2} \vec{f}\, d\vec{r} = \int_0^1 a^2\left(-\frac{(1+t)^2}{4}, \frac{(1+t)^2 - (2-t)^2}{4}, \frac{(2-t)^2}{4}\right) a\left(-\tfrac{1}{2}, 0, \tfrac{1}{2}\right) dt =$$
$$= \frac{a^3}{8}\int_0^1 [5 - 2t + 2t^2]\, dt = \frac{a^3}{8}\left(5 - 1 + \frac{2}{3}\right) = \frac{7a^3}{12}$$

$$\vec{r}_2(t) = a\left(\tfrac{1-t}{2}, \tfrac{t}{2}, 1\right) \longrightarrow \dot{\vec{r}}_2(t) = a\left(-\tfrac{1}{2}, \tfrac{1}{2}, 0\right)$$

kasneje enako 0

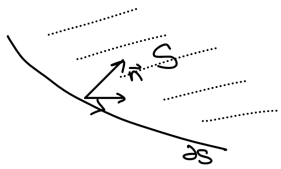
$$I_2 = \int_{T_2T_3} \vec{f}\, d\vec{r} = \int_0^1 a^3\left(\frac{t^2-4}{4}, \frac{4-(1-t)^2}{4}, \ldots\right)\left(-\tfrac{1}{2}, \tfrac{1}{2}, 0\right) dt = \frac{a^3}{8}\int_0^1 [7 + 2t - 2t^2]\, dt = \frac{11a^3}{12}.$$

HIPOTEZA: Ostali integrali so "simetrični": $\Rightarrow I = 3(I_1 + I_2) = \frac{9}{2}a^3$

I izračunamo s Stokesovim izrekom: $K = \partial S$; $S \ldots$ 6-kotnik

STOKESOV IZREK: $\int_{\partial S} \vec{f}\, d\vec{r} = \int_S \operatorname{rot}\vec{f}\, d\vec{S}$

usklajeni orientaciji:



$\operatorname{rot}\vec{f} = (-2y-2z,\ -2x-2z,\ -2x-2y)$

$d\vec{S} = \vec{n}\, dS$

enotska normala

S je del ravnine z normalo $(1,1,1)$; smer normale za Stokesov izrek je nasprotna $\Rightarrow$ $\vec{n} = \frac{-1}{\sqrt{3}}(1,1,1)$

$$I = \int_S \operatorname{rot}\vec{f}\cdot\vec{n}\, dS = \frac{2}{\sqrt{3}}\int_S (y+z,\ x+z,\ x+y)(1,1,1)\, dS =$$

$$= \frac{4}{\sqrt{3}}\int_S [x+y+z]\, dS$$

Koaj je $\int_S x\, dS$?

Če je S homogeno: $x_T = \dfrac{\int_S x\, dS}{\underbrace{\int_S dS}_{P(S)}}$

$$\Rightarrow \underline{\int_S x\, dS = x_T \cdot P(S)}$$

S je pravilni 6-kotnik s stranico $d = \frac{a}{\sqrt{2}}$

težišče 6-kotnika: $T(\underbrace{x_T, y_T, z_T}_{\text{enake koordinate}})$ $\Rightarrow 3x_T = \frac{3a}{2}$ $\Rightarrow T\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

$$P(S) = 6\cdot\frac{d^2\sqrt{3}}{4} = \frac{3}{2}\,\frac{a^2\sqrt{3}}{2} = \frac{3\sqrt{3}\,a^2}{4}$$

$\Rightarrow I = \frac{4}{\sqrt{3}}(x_T + y_T + z_T)P(S) = \frac{4}{\sqrt{3}}\cdot\frac{3}{2}a\ \frac{3\sqrt{3}}{4}\cdot a^2 = \frac{9}{2}a^3$ $\Leftarrow$ enak rezultat kot v hipotezi

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9. Dani sta števili $a > 0$ in α ter krivulja $K = S(0,a) \cap \Pi$, kjer je $S(a,0)$ sfera s središčem v $(0,0,0)$ in polmerom a ter Π ravnina z enačbo $y = x\,\mathrm{tg}\alpha$. Izračunaj integral

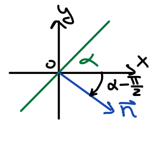
$$I = \int_K (y-z)dx + (z-x)dy + (x-y)dz$$

b) s pomočjo Stokesovega izreka.

$$I = \int_{\partial S} \vec{f}\, d\vec{r} = \int_S \operatorname{rot}\vec{f}\, d\vec{S}$$

$S \ldots$ krožnica, ki z x-osjo v xy-ravnini oklepa kot α

$\Rightarrow$ normala leži v xy-ravnini



$\Rightarrow \vec{n} = (\cos(\alpha - \frac{\pi}{2}), \sin(\alpha - \frac{\pi}{2}), 0) = (\sin\alpha, -\cos\alpha, 0)$

$\operatorname{rot}\vec{f} = (-1-1, -1-1, -1-1) = -2(1,1,1)$

enako kot pri direktnem izračunu

$$\Rightarrow I = \int_S \operatorname{rot}\vec{f}\cdot\vec{n}\, dS = -2\int_S (\sin\alpha - \cos\alpha)\, dS = 2(\cos\alpha - \sin\alpha)\underbrace{\int_S dS}_{P(S)} = 2(\cos\alpha - \sin\alpha)\pi a^2$$

$d\vec{S} = \vec{n}\, dS$

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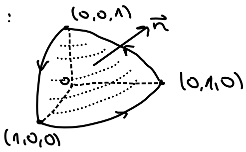
10. Naj bo $\vec{c} = (p, q, r)$ in $\vec{f}(\vec{r}) = \vec{c} \times \vec{r}$. Izračunaj cirkulacijo polja $\vec{f}$ vzdolž roba ploskve $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1,\ x, y, z \geq 0\}$.

del sfere v I. oktantu prostora:



Vemo: $\operatorname{rot}(\vec{c}\times\vec{r}) = 2\vec{c}$

$$\int_{\partial S} \vec{f}\, d\vec{r} = \int_S \operatorname{rot}\vec{f}\, d\vec{S} = I$$

Parametrizacija: $\vec{r}(\varphi, \vartheta) = (\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, \sin\vartheta)$

$d\vec{S} = (\vec{r}_\varphi \times \vec{r}_\vartheta)\, d\varphi\, d\vartheta$

$\vec{r}_\varphi = (-\sin\varphi\cos\vartheta, \cos\varphi\cos\vartheta, 0)$
$\vec{r}_\vartheta = (-\cos\varphi\sin\vartheta, -\sin\varphi\sin\vartheta, \cos\vartheta)$

$$\vec{r}_\varphi \times \vec{r}_\vartheta = (\cos\varphi\cos^2\vartheta, \sin\varphi\cos^2\vartheta, \sin^2\varphi\sin\vartheta\cos\vartheta + \cos^2\varphi\sin\vartheta\cos\vartheta) =$$
$$= \cos\vartheta(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, \sin\vartheta) =$$
$$= \cos\vartheta\, \vec{r}(\varphi, \vartheta)$$

(prava orientacija) $\quad \vec{c} = (p,q,r)$

$$I = \int_{D \leftarrow \text{območje za } (\varphi,\vartheta)} 2\vec{c}\cos\vartheta\, \vec{r}(\varphi, \vartheta)\, d\varphi\, d\vartheta = 2\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\vartheta\,(p\cos\varphi\cos\vartheta + q\sin\varphi\cos\vartheta + r\sin\vartheta)\, d\vartheta = (*)$$

$$\underbrace{\int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \int_0^{\frac{\pi}{2}} \sin^2 x\, dx}_{+ \Rightarrow \frac{\pi}{2}} = \frac{\pi}{4} \qquad \int_0^{\frac{\pi}{2}} \sin x \cos x\, dx = \frac{1}{2}$$

$$(*) = 2\int_0^{\frac{\pi}{2}} \left(p\frac{\pi}{4}\cos\varphi + q\frac{\pi}{4}\sin\varphi + r\frac{1}{2}\right) d\varphi = 2\left(p\frac{\pi}{4} + q\frac{\pi}{4} + r\frac{\pi}{4}\right) = \frac{\pi}{2}(p+q+r)$$

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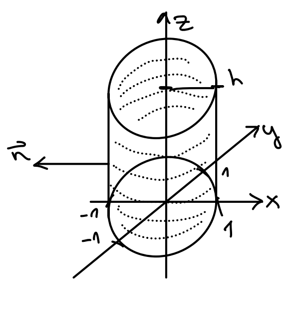
11. Naj bo S zunanja stran plašča valja, podanega z zvezama $x^2 + y^2 = 1$ in $z \in [0, h]$, kjer je $h > 0$. Izračunaj integral vektorskega polja (PRETOK)

$$\vec{f}(x, y, z) = (x + y, y + z, z + x)$$

po ploskvi S

a) direktno.

b) s pomočjo Gaussovega izreka.



$\vec{r}(\varphi, z) = (\cos\varphi, \sin\varphi, z)$ ← valjne koordinate
$\varphi \in [0, 2\pi],\ z \in [0, h]$

$\vec{r}_\varphi = (-\sin\varphi, \cos\varphi, 0)$
$\vec{r}_z = (0, 0, 1)$

$\vec{r}_\varphi \times \vec{r}_z = (\cos\varphi, \sin\varphi, 0)$

kaže v pravo smer

(a) $$\int_S \vec f\, d\vec S = \int_0^{2\pi} d\varphi \int_0^h (\cos\varphi+\sin\varphi,\ \sin\varphi+z,\ z+\cos\varphi)(\cos\varphi, \sin\varphi, 0)\, dz =$$

$$= \int_0^{2\pi} d\varphi \int_0^h \Big(1 + \underbrace{\cos\varphi\sin\varphi}_{\frac12 \sin(2\varphi)} + z\sin\varphi\Big)\, dz = \qquad \int_0^{2\pi} \sin 2\varphi\, d\varphi = 0 = \int_0^{2\pi} \sin\varphi\, d\varphi$$

$$= 2\pi h$$

(b) zunanja normala $\quad$ GAUSSOV IZREK

$$\boxed{\int_{\partial D} \vec f\, d\vec S = \int_D \operatorname{div} \vec f\, dV} = 3\int_D dV = 3V(D) = 3\pi h = \int_S \dots + \int_{K_0} \dots + \int_{K_h} \dots$$

$\operatorname{div} \vec f = 1+1+1 = 3 \qquad D = \text{valj} = K(0,1) \times [0,h]$

$$\partial D = \underset{\text{plašč}}{S} \cup \overset{\text{spodnja}}{K_0} \cup \underset{\text{zgornja}}{K_h}$$

$$\Rightarrow \int_S \dots = 3\pi h - \int_{K_0} \dots - \int_{K_h} \dots = 3\pi h - \pi h = 2\pi h$$

$$\int_{K_h} \vec f\, d\vec S = \int_{K_h} \vec f\,(0,0,1)\, dS = \int_{K_h} (z+x)\, dS \overset{z=h}{=} \int_{K_h} (h+x)\, dS = \int_{K_h} h\, dS + \int_{K_h} x\, dS = h\cdot P(K_h) + \overset{0}{x_T} \cdot P(K_h) = h\pi$$

$d\vec S = \vec n\, dS$

$$\Rightarrow \int_{K_0} \vec f\, d\vec S = 0$$ normala $(0,0,-1)$

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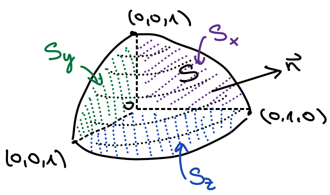
12. Naj bo $S = \{(x,y,z) \mid x^2+y^2+z^2 = 1,\ x,y,z \geq 0\}$. Izračunaj integral

$$\int_S xy\, dz dy + yz\, dx dz + xz\, dx dy$$

a) direktno. (DN)

b) s pomočjo Gaussovega izreka.

(b):



$D \dots$ krogla v I. oktantu

$\partial D = S \cup S_y \cup S_x \cup S_z$

$\vec f = (xy,\ yz,\ xz)$

$\operatorname{div} \vec f = y + x + z$

sferične koordinate

$$\int_D \operatorname{div} \vec f\, dV = \int_D (y+x+z)\, dV = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\vartheta \int_0^1 r(\cos\varphi\cos\vartheta + \sin\varphi\cos\vartheta + \sin\vartheta)\, r^2 \cos\vartheta\, dr =$$

$$= \frac14 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \Big[\cos\varphi \underbrace{\cos^2\vartheta}_{\frac{\pi}{4}} + \sin\varphi \underbrace{\cos^2\vartheta}_{\frac{\pi}{4}} + \underbrace{\sin\vartheta\cos\vartheta}_{\frac12}\Big]\, d\vartheta =$$

$$= \frac14 \int_0^{\frac{\pi}{2}} \Big[\frac{\pi}{4}\cos\varphi + \frac{\pi}{4}\sin\varphi + \frac12\Big]\, d\varphi =$$

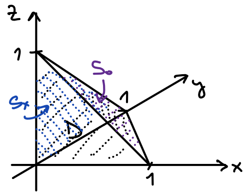
$$= \frac{\pi}{16} + \frac{\pi}{16} + \frac{\pi}{16} =$$

$$= \frac{3\pi}{16}$$

13. Izračunaj integral

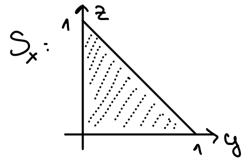
$$\int_{\partial D} \frac{dS}{(1+x+y)^2},$$

kjer je $D = \{(x,y,z) \mid x+y+z \le 1,\ x,y,z \ge 0\}$.



$\partial D = S_x \cup S_y \cup S_z \cup S_0$

S_x:



Parametrizacija območja v ravnini $x=a$: $\vec r(y,z) = (a,y,z)$

Splošno: $dS = |\vec r_u \times \vec r_v|\,du\,dv$; $|\vec r_u \times \vec r_v| = \sqrt{EG - F^2}$ $E = \vec r_u \cdot \vec r_u$ $F = \vec r_u \cdot \vec r_v$, $G = \vec r_v \vec r_v$

$\Rightarrow \underline{dS} = |(0,1,0)\times(0,0,1)|\,dy\,dz = |(1,0,0)|\,dy\,dz = \underline{dy\,dz}$

$$\underline{I_x} = \int_0^1 dy \int_0^{1-y} \frac{1}{(1+\underset{x=0}{x}+y)^2}\,dz = \int_0^1 dy \int_0^{1-y} \frac{dz}{(1+y)^2} = \int_0^1 \frac{1-y}{(1+y)^2}\,dy \overset{t=1+y}{=} \int_1^2 \frac{2-t}{t^2}\,dt = \left(-\frac{2}{t} - \ln|t|\right)\Big|_1^2 = \underline{1-\ln 2}$$

enaka integrala

$$\underline{I_y} = \int_0^1 dx \int_0^{1-x} \frac{dz}{(1+x)^2} = \underline{1 - \ln 2}$$

$$\underline{I_z} = \int_0^1 dx \int_0^{1-x} \frac{dy}{(1+x+y)^2} = \int_0^1 \left[-(1+x+y)^{-1}\right]_0^{1-x} dx = \int_0^1 \left[-\frac{1}{2} + \frac{1}{1+x}\right] dx = -\frac{x}{2} + \ln(1+x)\Big|_0^1 = \underline{-\frac{1}{2} + \ln 2}$$

$$\Bigg\}\ \int f(y)\,dy = F(y) + c \Rightarrow \int f(y+a)\,dy = F(y+a) + c$$

S_0: $\vec r(u,v) = (1,0,0) + u(-1,0,1) + v(-1,1,0) = (1-u-v, v, u)$

$dS = |(-1,0,1)\times(-1,1,0)|\,du\,dv = \sqrt{3}\,du\,dv$

Določanje meje: $x,y,z \in [0,1] \Leftrightarrow (1-u-v, v, u) \in [0,1]^3$

$0 \le 1-u-v \le 1$

$u - 1 \le -v \le u \Rightarrow 1-u \ge v \ge -u$ (z $-u \le 0$) $\Rightarrow v \in [0, 1-u]$

$$\underline{I_0} = \int_0^1 du \int_0^{1-u} \frac{\sqrt{3}\,dv}{(2-u)^2} = \int_0^1 \sqrt{3}\,\frac{1-u}{(2-u)^2}\,du \overset{t=2-u}{=} \sqrt{3}\int_2^1 -\frac{t-1}{t^2}\,dt = \sqrt{3}\int_1^2 \left[\frac{1}{t} - \frac{1}{t^2}\right] dt = \sqrt{3}\left[\ln t + \frac{1}{t}\right]_1^2 = \underline{\sqrt{3}\left(\ln 2 - \frac{1}{2}\right)}$$

Opomba: Parametrizacija S_0 "po receptu": $x+y+z=1 \Rightarrow x = 1-y-z$
$\Rightarrow \vec r(y,z) = (1-y-z, y, z)$; (y,z) leži v S_x $\xrightarrow{\text{meje}} \int_0^1 dy \int_0^{1-y} dz$

$$\Rightarrow \underline{I} = I_x + I_y + I_z + I_0 = 1 - \ln 2 + 1 - \ln 2 - \frac{1}{2} + \ln 2 + \sqrt{3}\ln 2 - \frac{\sqrt{3}}{2} = \underline{(\sqrt{3}-1)\ln 2 + \frac{3-\sqrt{3}}{2}}$$

17. Izračunaj integral

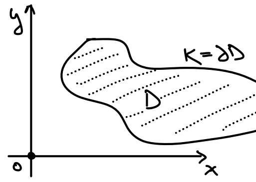
$$\int_K \frac{x\,dy - y\,dx}{x^2+y^2}, \qquad (K = \partial D)$$

če je $K \subset \mathbf{R}^2$ zaključena krivulja,

a) ki ne obkroži izhodišča.

b) ki obkroži izhodišče.

(a)



K ne obkroži izhodišča $\Leftrightarrow (0,0) \notin D$

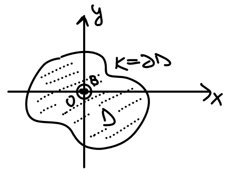
$(0,0) \notin K$ privzamemo

$$\int_{\partial D} P\,dx + Q\,dy \overset{\text{Green}}{=} \iint_D (Q_x - P_y)\,dx\,dy$$

$$I = \int_{\partial D} \underbrace{\frac{-y}{x^2+y^2}}_{P}\,dx + \underbrace{\frac{x}{x^2+y^2}}_{Q}\,dy =$$

$$= \iint_D \left(\overset{Q_x}{\frac{y^2-x^2}{(x^2+y^2)^2}} - \overset{P_y}{\frac{y^2-x^2}{(x^2+y^2)^2}} \right) dx\,dy = 0$$

(b)



Greenove formule ne moremo uporabiti, ker $\vec{f} = (P,Q)$ v $(0,0)$ ni definirano.

Iz D izrežemo $K((0,0),r) = B$, $B \subseteq \mathrm{Int}\,D$.

Definiramo $D' = D \setminus B$

$\Rightarrow$ na D' lahko uporabimo Greenovo formulo:

$$\Rightarrow \int_K P\,dx + Q\,dy \overset{\text{Green}}{=} 0 \quad \text{kot prej}$$

$$\partial D' = \partial D \cup \partial B^-$$

Rob okoli "luknje" B orientiramo v obratno smer: ∂B^-.

$$0 = \int_{\partial D'} \dots = \int_{\partial D} \dots + \int_{\partial B^-} \dots = \int_{\partial D} \dots - \int_{\partial B} \quad \Rightarrow \int_{\partial D} = \int_{\partial B}$$

$$\int_{\partial B} \frac{x\,dy - y\,dx}{x^2+y^2} = I: \quad \text{Parametrizacija } \partial B: \quad x = r\cos\varphi \quad y = r\sin\varphi \; ; \; \varphi \in [0, 2\pi]$$

$$dx = -r\sin\varphi \qquad dy = r\cos\varphi$$

$$I = \int_0^{2\pi} \frac{r^2\cos^2\varphi + r^2\sin^2\varphi}{r^2\cos^2\varphi + r^2\sin^2\varphi}\,d\varphi = 2\pi$$

$$\Rightarrow \int_{\partial D} \frac{x\,dy - y\,dx}{x^2+y^2} = 2\pi$$

14. Dana so števila $m, n, p > 0$ in a, b, c. Izračunaj integral

$$\int_S x^2\,dzdy + y^2\,dxdz + z^2\,dxdy,$$

kjer je S zunanja stran elipsoida z enačbo

$$\left(\frac{x-a}{m}\right)^2 + \left(\frac{y-b}{n}\right)^2 + \left(\frac{z-c}{p}\right)^2 = 1.$$ ← središče (a,b,c)

polosi: m,n,p

$I=\int_S \vec{f}\,d\vec{S}$; $\vec{f}=(x^2,y^2,z^2)$

Parametrizacija S:
$$\left.\begin{aligned} x &= m\cos\varphi\cos\vartheta + a\\ y &= n\cdot\sin\varphi\cos\vartheta + b\\ z &= p\cdot\sin\vartheta + c\end{aligned}\right\}\ \vec{r}(\varphi,\vartheta) \qquad \varphi\in[0,2\pi],\ \vartheta\in\left[-\tfrac{\pi}{2},\tfrac{\pi}{2}\right]$$

$$\begin{aligned} d\vec{S} &= |\vec{r}_\varphi\times\vec{r}_\vartheta|\,d\varphi\,d\vartheta =\\ &= |(-m\sin\varphi\cos\vartheta,\ n\cos\varphi\cos\vartheta,\ 0)\times(-m\cos\varphi\sin\vartheta,\ -n\sin\varphi\sin\vartheta,\ p\cos\vartheta)|\,d\varphi\,d\vartheta =\\ &= |(np\cos\varphi\cos^2\vartheta,\ mp\sin\varphi\cos^2\vartheta,\ mn\sin^2\varphi\cos\vartheta\sin\vartheta + mn\cos^2\varphi\cos\vartheta\sin\vartheta)|\,d\varphi\,d\vartheta\end{aligned}$$

Orientacija: normala $\vec{r}_\varphi\times\vec{r}_\vartheta$ pri $\varphi=0$, $\vartheta=0$: $(np,0,0)$

$$I=\int_0^{2\pi}d\varphi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left((m\cos\varphi\cos\vartheta+a)^2,(n\sin\varphi\cos\vartheta+b)^2,(p\sin\vartheta+c)^2\right)\cdot(np\cos\varphi\cos^2\vartheta,\ mp\sin\varphi\cos^2\vartheta,\ mn\sin\vartheta\cos\vartheta)\,d\vartheta$$

$$= \dots = \dots$$

$$\begin{cases}\displaystyle\int_0^{2\pi}\cos^{2k+1}\varphi\,(\dots)\,d\varphi = 0, \quad \int_0^{2\pi}\sin^{2k+1}\varphi\,(\dots)\,d\varphi = 0\\ \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2k+1}\vartheta\,(\dots)\,d\vartheta = 0\end{cases}$$

2. način: (Gaussov izrek)

$\operatorname{div}\vec{f} = 2(x+y+z)$

$S=\partial D$:
$$I=\int_S\vec{f}\,d\vec{S}=\int_D\operatorname{div}\vec{f}\,dV = 2\int_D(x+y+z)\,dV = \underbrace{2\int_D x\,dV + 2\int_D y\,dV + 2\int_D z\,dV}_{\text{koordinate težišča}\,\cdot\,V(D)\,\times 2} \overset{(*)}{=} 2(a+b+c)\frac{4\pi}{3}mnp$$

$V(D)=\frac{4\pi}{3}mnp$ (*) $\qquad\Rightarrow\ I=\frac{8}{3}\pi(a+b+c)mnp$

⇑ raztegi

$V_{krogla}=\frac{4\pi r^3}{3}$

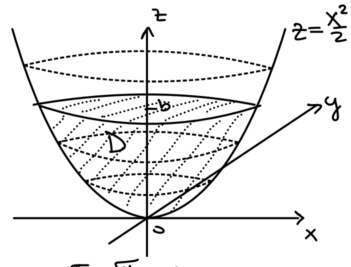
---

15. Dano je vektorsko polje $\vec{f}(\vec{r}) = |\vec{r}|^2\vec{r}$ in število $b > 0$. Izračunaj pretok polja $\vec{f}$ skozi

a) rob območja $D = \{(x,y,z) \mid 2z \geq x^2+y^2,\ z \leq b\}$.

b) ploskev podano z zvezama $2z = x^2+y^2$ in $z \leq b$.

(a)

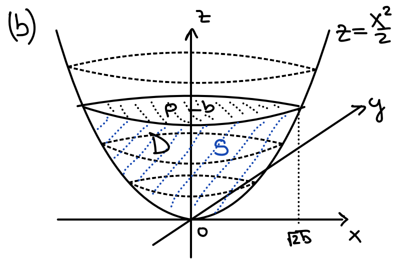


$$\int_{\partial D}\vec{f}\,d\vec{S} = \iiint_D\operatorname{div}\vec{f}\,dV = \qquad \vec{f}=(x^2+y^2+z^2)(x,y,z) \qquad \operatorname{div}\vec{f} = 5x^2+5y^2+5z^2$$

$$= 5\iiint_D(x^2+y^2+z^2)\,dV = \quad \leftarrow \text{Valjne koordinate} \quad \left.\begin{aligned}x&=r\cos\varphi\\ y&=r\sin\varphi\\ z&=z\end{aligned}\right\}\ |\det JF| = r$$

$$= 5\int_0^{2\pi}d\varphi\int_0^{\sqrt{2b}}dr\int_{\frac{r^2}{2}}^{b}(r^2+z^2)r\,dz = 10\pi\int_0^{\sqrt{2b}}r\left(r^2b-\frac{r^4}{2}+\frac{b^3}{3}-\frac{r^6}{6}\right)dr = 10\pi\left(\frac{b^3}{3}+\frac{b^4}{4}\right)$$

(b)



$\partial D = S \cup P$

$$\iint_S \vec{f}\, d\vec{S} = 10\pi\left(\frac{b^3}{3}+\frac{b^4}{4}\right) - \iint_P \vec{f}\, d\vec{S} = 10\pi\left(\frac{b^3}{3}+\frac{b^4}{4}\right) - 2\pi\left(b^3+b^4\right) =$$
$$= \underline{2\pi\left(\frac{2}{3}b^3+\frac{1}{4}b^4\right)}$$

$$\iint_P \vec{f}\, d\vec{S} = \int_P \vec{f}\cdot \underset{(0,0,1)}{\vec{n}}\, dS \overset{z=b}{=} \int_0^{2\pi} d\varphi \int_0^{\sqrt{2b}} r b(r^2+b^2)\, dr = 2\pi\left(\frac{b}{4}\sqrt{2b}^4 + \frac{b^3}{2}\sqrt{2b}^2\right) = 2\pi\left(b^3+b^4\right)$$

$$\vec{f}\cdot n = |\vec{r}|^2\vec{r}\cdot\vec{n} = (x^2+y^2+z^2)(x,y,z)(0,0,1) = (x^2+y^2+z^2)z = (r^2+z^2)z$$

valjne koordinate

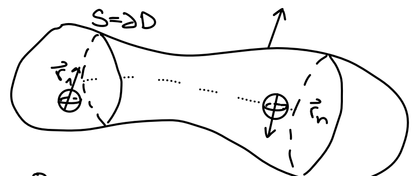
---

16. Dane so točke $\vec{r}_1, \dots, \vec{r}_n \in \mathbf{R}^3$ in števila $e_1, \dots e_n$. Izračunaj pretok polja

$$\vec{F}(\vec{r}) = \sum_{i=1}^{n} \operatorname{grad} \frac{-e_i}{4\pi|\vec{r}-\vec{r}_i|}$$

skozi zaključeno ploskev, ki objame vse točke $\vec{r}_1, \dots, \vec{r}_n$.

$S=\partial D$: S objema $\vec{r}_1,\dots,\vec{r}_n \Leftrightarrow \vec{r}_1,\dots,\vec{r}_n \in \operatorname{Int} D$



Vemo: $\operatorname{grad}\frac{1}{|\vec{r}-\vec{a}|} = -\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|}$ in $\operatorname{div}\left(\operatorname{grad}\frac{1}{|\vec{r}-\vec{a}|}\right) = 0$

Iz D izrežemo kroglice $K_i = K(\vec{r}_i, \varepsilon)$, da $\overline{K_i} \subseteq \operatorname{Int} D$ in $\overline{K_i} \cap \overline{K_j} = \emptyset$ za $i \neq j$. (*)

$\rightarrow D' = D \setminus \bigcup_{i=1}^{n} K_i \longrightarrow \partial D' = \underset{\text{zunanja normala}}{\partial D^+} \cup \bigcup_{i=1}^{n} \partial K_i^-$ notranja normala

$$\iint_{\partial D'} \vec{F}\, d\vec{S} \overset{\text{Gauss}}{=} \iiint_{D'} \underbrace{\operatorname{div}\vec{F}}_{0}\, dV = 0$$

$$0 = \iint_{\partial D^+} \vec{F} d\vec{S} - \iint_{\bigcup_{i=1}^n \partial K_i} \vec{F} d\vec{S} \longrightarrow \iint_{\partial D^+} \vec{F} d\vec{S} = \iint_{\bigcup_{i=1}^n \partial K_i^-} \vec{F} d\vec{S}$$

$$\iint_{\partial K_j} \vec{F} d\vec{S} = \iint_{\partial K_j} \sum_{i=1}^{n} \frac{e_i(\vec{r}-\vec{r}_i)}{4\pi|\vec{r}-\vec{r}_i|^3} d\vec{S} = \sum_{i=1}^{n} \iint_{\partial K_j} \frac{e_i}{4\pi} \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} d\vec{S} = \sum_{i=1}^{n} \frac{e_i}{4\pi} \iint_{\partial K_j} \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} d\vec{S} =$$

$$= \underbrace{\sum_{i=1}^{n} \frac{e_i}{4\pi} \iiint_{K_j} \underbrace{\operatorname{div}\vec{F}_i}_{0}\, dV}_{i\neq j \;=\; 0} + \iint_{\partial K_j} \frac{e_j(\vec{r}-\vec{r}_j)}{4\pi|\vec{r}-\vec{r}_j|^3} d\vec{S} = \iint_{\partial K_j} \frac{e_j}{4\pi} \frac{\varepsilon\vec{n}}{\varepsilon^3}\vec{n}\, dS = \iint_{\partial K_j} \frac{e_j}{4\pi\varepsilon^2} dS = \frac{e_j}{4\pi\varepsilon^2}\cdot 4\pi\varepsilon^2 = e_j$$

$\vec{r} \in \partial K_j$: $d\vec{S} = \vec{n}\, dS$, $\vec{r}-\vec{r}_j = \varepsilon\vec{n}$

$$\underline{\iint_S \vec{F} d\vec{S}} = \iint_{\partial D} \vec{F} d\vec{S} \overset{(*)}{=} \sum_{j=1}^{n} \iint_{\partial K_j} \vec{F} d\vec{S} = \underline{\sum_{j=1}^{n} e_j}$$

---

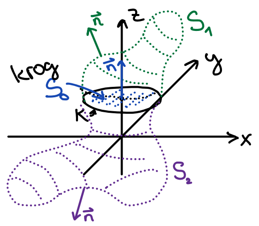
18. Dan je ploskovni integral

$$I = \int_S (1+x^2)f(x)\, dydz - 2xyf(x)\, dzdx - 3z\, dxdy.$$

Določi zvezno odvedljivo funkcijo f tako, da bo integral I enak za vse ploskve S, katerih rob je krožnica $\{(\cos t, \sin t, 1) \mid t \in [0, 2\pi]\}$ in tedaj izračunaj integral I.

$= K$

$$I=\iint_S \vec{F}\,d\vec{S}=\iint_S \left((1+x^2)f(x),\,-2xyf(x),\,-3z\right)d\vec{S}$$



$\partial S=K:\ \rightarrow \iint_{S_1}\vec{F}\,d\vec{S}=\iint_{S_2}\vec{F}\,d\vec{S}$ $\qquad \vec{n}_0=(0,0,1)$

$S_1\cup S_2$ zaključena ploskev

Privzemimo, da je $S_1\cup S_2=\partial D$
$\partial D^+=S_1^+\cup S_2^-$

$$\iint_{\partial D^+}\vec{F}\,d\vec{S}\overset{\text{Gauss}}{=}\iiint_D \operatorname{div}\vec{F}\,dV$$

$$=\iint_{S_1^+}\vec{F}\,d\vec{S}+\iint_{S_2^-}\vec{F}\,d\vec{S}=\underbrace{\iint_{S_1^+}\vec{F}\,d\vec{S}-\iint_{S_2^+}\vec{F}\,d\vec{S}}_{=\text{(želja)}}=0$$

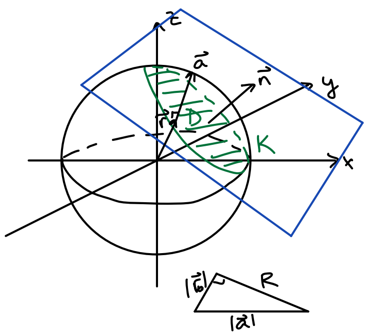
Če je $\operatorname{div}\vec{F}=0$, si "izpolnemo želje".

Vemo: $\operatorname{div}\left((1+x^2)f(x),-2xyf(x),-3z\right)=0 \Leftrightarrow f(x)=3\arctan x+C$

$$\Rightarrow \underline{\iint_S \vec{F}\,d\vec{S}}=\iint_{S_0}\vec{F}\,d\vec{S}=\iint_{S_0}\vec{F}\cdot\vec{n}_0\,dS=\iint_{S_0}\vec{F}\cdot(0,0,1)\,dS=-3\iint_{S_0}z\,dS\overset{z=1\text{ na }S_0}{=}-3\cdot P(S_0)\overset{\pi\cdot r^2=\pi}{=}\underline{-3\pi}$$

---

19. Dana sta vektorja $\vec{a}\neq 0$, $\vec{b}$ in vektorsko polje $\vec{f}(\vec{r})=(\vec{r}-\vec{a})\times(\vec{r}-\vec{b})$. Izračunaj cirkulacijo polja $\vec{f}$ vzdolž krivulje, ki jo podajata zvezi $|\vec{r}|=|\vec{a}|$ in $\vec{r}\cdot\vec{a}=\frac{|\vec{a}|^2}{2}$.



$\vec{r}\cdot\vec{a}=D$ … ravnina z normalo $\vec{a}$ skozi točko $\vec{r}_0=\frac{\vec{a}}{2}$

$$I=\int_K \vec{F}\,d\vec{r}=\int_{\partial D}\vec{F}\,d\vec{r}\overset{\text{Stokes}}{=}\iint_D \operatorname{rot}\vec{F}\,d\vec{S}=(*)$$

$$\vec{F}=\vec{r}\times\vec{r}-\vec{r}\times\vec{b}-\vec{a}\times\vec{r}+\vec{a}\times\vec{b}=\vec{r}\times(\vec{a}-\vec{b})+\vec{a}\times\vec{b}$$

Vemo: $\operatorname{rot}(\vec{r}\times\vec{c})=-2\vec{c}$ $\qquad\rightarrow \operatorname{rot}\vec{F}=-2(\vec{a}-\vec{b})+\underbrace{\operatorname{rot}(\vec{a}\times\vec{b})}_{0}=2(\vec{b}-\vec{a})$

$d\vec{S}=\vec{n}\,dS=\frac{\vec{a}}{|\vec{a}|}dS$

$$(*)=\iint_D 2(\vec{b}-\vec{a})\frac{\vec{a}}{|\vec{a}|}dS=2(\vec{b}-\vec{a})\frac{\vec{a}}{|\vec{a}|}\iint_D dS=2(\vec{b}-\vec{a})\frac{\vec{a}}{|\vec{a}|}\cdot P(D)\overset{\text{krog s polmerom }R}{=}2(\vec{b}-\vec{a})\frac{\vec{a}}{|\vec{a}|}\pi R^2$$

$$\Rightarrow \underline{I}=2(\vec{b}-\vec{a})\frac{\vec{a}}{|\vec{a}|}\pi\frac{3}{4}|\vec{a}|^2=\underline{(\vec{b}-\vec{a})\vec{a}\pi\cdot\frac{3}{2}|\vec{a}|}$$

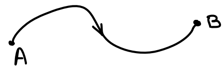
---

22. Dokaži, da je $\vec{f}(x,y,z)=(2x\cos y-y^2\sin x,2y\cos x-x^2\sin y,4)$ potencialno, določi njegov potencial in izračunaj integral polja $\vec{f}$ vzdolž krivulje s parametrizacijo $\vec{r}(t)=(\cos t,\sin t,t)$, kjer je $t\in[-2\pi,2\pi]$.

$\vec{f}$ je potencialno $\Leftrightarrow \vec{f}=\operatorname{grad}u$; u potencial $\vec{f}$

Vemo: $\underline{u=x^2\cos y+y^2\cos x+4z+C}$

$$\int_K \vec f\, d\vec r = u(B)-u(A)$$



$\vec r(-2\pi)=A$, $\vec r(2\pi)=B$

$$\to A(1,0,-2\pi),\ B(1,0,2\pi) \to \int_K \vec f(x,y,z)\,dr = u(1,0,2\pi)-u(1,0,-2\pi) = 16\pi$$

---

23. Določi konstanti a in b tako, da bo polje

$$\vec f(x,y,z) = (2(axzy^4 - y), 2(bx^2zy^3 - x), 3x^2y^4)$$

potencialno in izračunaj integral $\int_K \vec f d\vec r$, kjer je K krivulja z začetno točko $(0,0,0)$ in končno točko $(1,1,1)$ ter pretok skozi površje kocke $[0,1]^3$.

Od prej: $a=3,\ b=6$

$u = 3x^2y^4z - 2xy + C$

$$\to \int_K \vec f\, d\vec r = u(1,1,1) - u(0,0,0) = 1$$

---

21. Naj bo K zaključena krivulja, ki poteka po sferi $x^2+y^2+z^2=1$. Izračunaj integral

$$I = \int_K \frac{dx+dy+dz}{\underbrace{(x^2+y^2+z^2)^2}_{|\vec r|^2}}.$$

$\vec r \in K: |\vec r| = 1$

K zaključena krivulja (z=K)

$$I = \int_K \vec F\, d\vec r = \int_K \underbrace{\frac{1}{|\vec r|^4}(1,1,1)}_{\vec F} d\vec r = \int_K \underbrace{(1,1,1)}_{\vec a} d\vec r = 0$$

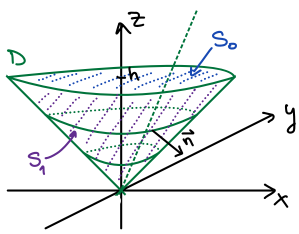
$\vec F$ ← ni potencialno

$\vec a$ ← je potencialno na K $\to u_{\vec a} = x+y+z$

enaka na K (ne nujno povsod)

---

20. Naj bo $h>0$ in $\vec f(\vec r) = \vec r$. Izračunaj pretok polja $\vec f$ skozi plašč in osnovno ploskev stožca, podanega z zvezama $x^2+y^2 \le z^2$ in $0 \le z \le h$.



$$\iint_{S_1} \vec f\, d\vec S = \iint_{S_1} \vec f\cdot\vec n\, dS = \iint_1 \vec 0\, dS = 0$$ ← plašč

$$\int_{\partial D} = \int_{S_0} + \underbrace{\int_{S_1}}_{0}$$

stožec: $\frac{1}{3}\cdot h\cdot\pi r^2$

osnovna ploskev

$$\iint_{S_0} \vec f\, d\vec S = \iint_{\partial D} \vec f\, d\vec S = \iiint_D 3\,dV = 3\cdot V(D) = 3\cdot\frac{1}{3} h\cdot\pi h^2 = \pi h^3$$

24. Dani sta dvakrat zvezno odvedljivi skalarni polji u in v in območje $D \subset \mathbf{R}^3$ s kosoma gladkim robom. Za enotski vektor $\vec{e}$ definiramo *smerni odvod* kot

$$\frac{\partial u}{\partial \vec{e}} = \operatorname{grad} u \cdot \vec{e}.$$

Naj $\vec{n}$ označuje enotsko normalo ploskve ∂D. Dokaži:

a)

$$\int_{\partial D} u \frac{\partial v}{\partial \vec{n}}\, dS = \int_D (\operatorname{grad} u \cdot \operatorname{grad} v + u \triangle v) dV$$

b)

$$\int_{\partial D} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}} \right) dS = \int_D (u \triangle v - v \triangle u) dV$$

(a)
$$\iint_{\partial D} u \cdot \operatorname{grad} v \cdot \vec{n}\, dS = \iint_{\partial D} u \operatorname{grad} v\, d\vec{S} \overset{\text{Gauss}}{=} \iiint_D \operatorname{div}(u \operatorname{grad} v)\, dV = \iiint_D [\operatorname{grad} u \cdot \operatorname{grad} v + u \Delta v]\, dV$$

$$\operatorname{div}(u \cdot \operatorname{grad} v) = \operatorname{div}(u v_x, u v_y, u v_z) = u_x v_x + u v_{xx} + u_y v_y + u v_{yy} + u_z v_z + u v_{zz} =$$
$$= \operatorname{grad} u \cdot \operatorname{grad} v + u \cdot \underbrace{\Delta v}_{= v_{xx} + v_{yy} + v_{zz}}$$

(b)
$$\iint_{\partial D} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\right) dS = \iint_{\partial D} u \frac{\partial v}{\partial \vec{n}}\, dS - \iint_{\partial D} v \frac{\partial u}{\partial \vec{n}}\, dS \overset{\text{Gauss}}{=} \iiint_D (\operatorname{grad} u \cdot \operatorname{grad} v + u \Delta v)\, dV -$$
$$- \iiint_D (\operatorname{grad} v \cdot \operatorname{grad} u + v \Delta u)\, dV =$$
$$= \iiint_D (u \Delta v - v \Delta u)\, dV$$

25. Naj bo $K \subset (0, \infty) \times (0, \infty)$ neka krivulja med točkama $(1, 1)$ in $(2, 2)$. Določi funkcijo u tako, da bo integral $\int_K u(x, y)(y\, dx + x\, dy)$ neodvisen od izbire krivulje K.

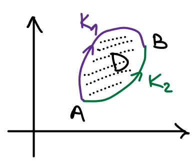
Namig: Novi koordinati $t = xy$ in $s = \frac{x}{y}$.

$\int_K \underbrace{P dx + Q dy}_{(P,Q)\, d\vec{r}}$ je neodvisen od K, če je (P, Q) potencialno $\Leftrightarrow$ $\overbrace{(P, Q, 0)}^{\vec{f}}$ potencialno $\Leftrightarrow \operatorname{rot}(P, Q, 0) = \vec{0}$

$\vec{f}(x, y, z) = (P(x, y), Q(x, y), 0)$ na $\underbrace{(0, \infty)^2 \times \mathbb{R}}_{\text{konveksna množica}}$

$\operatorname{rot} \vec{f} = (0, 0, \underbrace{Q_x - P_y}_{= 0})$ (Gauss)

Opomba: Do pogoja $Q_x - P_y = 0$ lahko pridemo tudi:



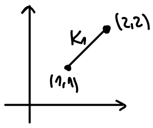
$$\int_{\partial D} = \underbrace{\int_{K_2} - \int_{K_1}}_{= 0 \text{ (želja)}} \overset{\text{Green}}{=} \int_D (Q_x - P_y)\, dV \Rightarrow (Q_x - P_y) = 0$$

$$\left.\begin{array}{l} P = y u(x, y) \\ Q = x \cdot u(x, y) \end{array}\right\} \Rightarrow Q_x - P_y = x u_x(x, y) - y u_y(x, y) \quad (+ u(x, y) - u(x, y))$$
$$= x u_x - y u_y$$

$t = xy,\ s = \frac{x}{y}$:
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x} = y \frac{\partial}{\partial t} + \frac{1}{y}\frac{\partial}{\partial s}$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial}{\partial s}\frac{\partial s}{\partial y} = x \frac{\partial}{\partial t} - \frac{x}{y^2}\frac{\partial}{\partial s}$$

$$\rightarrow x u_x - y u_y = x\left(y u_t + \frac{1}{y} u_s\right) - y\left(x u_t - \frac{x}{y^2} u_s\right) = \frac{2x}{y} u_s = 0 \rightarrow u_s = 0 \Leftrightarrow u \text{ odvisen le od } t$$
$$\Rightarrow u = g(t) = g(xy)$$

$$I = \iint_K y g(xy)\,dx + x g(xy)\,dy$$



$K_1: t \mapsto (t,t) \Rightarrow I = \int_{(1,1)}^{(2,2)} t g(t^2)\,dt + t g(t^2)\,dt = 2\int_{(1,1)}^{(2,2)} t g(t^2)\,dt$

Če je G primitivna funkcija za g: $G' = g$, potem $2tg(t^2) = (H(t))'$; $H(t) = G(t^2)$

$$\Rightarrow I = \int_1^2 G(t^2)'\,dt = G(4) - G(1)$$

HOLOMORFNE FUNKCIJE

$D^{odp} \subseteq \mathbb{C}$ in $f: D \to \mathbb{C}$

f je holomorfna na D, če je odvedljiva v kompleksnem smislu v vsaki točki D

$\left(f: D \to \mathbb{C},\ \text{za } \forall z_0 \in D.\ \exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0)\right)$

$f(x+iy) = u(x,y) + iv(x,y) \in \mathcal{H}(D) \iff \underbrace{u_x = v_y \ \wedge\ u_y = -v_x}_{\text{CAUCHY-RIEMANNOV SISTEM}}$

---

(1) Naj bo $D \subset \mathbb{C}$ povezana odprta množica in f funkcija, holomorfna na D. Katere od funkcij f_1, f_2 in f_3, definiranih s predpisi $f_1(z) = \overline{f(z)}$, $f_2(z) = f(\overline{z})$ in $f_3(z) = \overline{f(\overline{z})}$, so holomorfne?

f_1: $f_1(x+iy) = \overline{f(x+iy)} = u(x,y) - iv(x,y)$

$f_1 = u_1 - iv_1$: $u_1(x,y) = u(x,y)$
$v_1(x,y) = -v(x,y)$

$(u_1)_x = u_x(x,y) = -v_y(x,y) = (v_1)_y$
$f \in \mathcal{H}(D)$: $u_x = v_y$
$\Rightarrow u_x = v_y = 0$

$(u_1)_y = u_y = v_x = -(v_1)_x$
$f \in \mathcal{H}(D)$: $u_y = -v_x$
$\Rightarrow u_y = v_x = 0$

$\Rightarrow$ u in v sta konstantni $\cancel{}$

$\Rightarrow f_1$ ni holomorfna

f_2: $f_2(x+iy) = f(\overline{x+iy}) = f(x-iy) = u(x,-y) + iv(x,-y)$

$f_2 = u_2 + iv_2$: $u_2(x,y) = u(x,-y)$
$v_2(x,y) = v(x,-y)$

$(u_2)_x = u_x(x,-y) \underset{f \in \mathcal{H}(D)}{=} v_y(x,y) = -(v_2)_y \Rightarrow u_x = v_y = 0$

$(u_2)_y = -u_y(x,-y) \underset{f \in \mathcal{H}(D)}{=} v_x(x,-y) = (v_2)_x \Rightarrow u_y = v_x = 0$

$\Rightarrow$ u in v konstantni $\cancel{}$

$\Rightarrow f_2$ ni holomorfna

f_3: $f_3(x+iy) = \overline{f_3(\overline{x+iy})} = u(x,-y) - iv(x,-y)$

$f_3 = u_3 - iv_3$: $u_3(x,y) = u(x,-y)$
$v_3(x,y) = -v(x,-y)$

$(u_3)_x = u_x(x,-y) \underset{f \in \mathcal{H}(D)}{=} v_y(x,-y) = (v_3)_y$ ✓

$(u_3)_y = -u_y(x,-y) \underset{f \in \mathcal{H}(D)}{=} v_x(x,-y) = -(v_3)_x$ ✓

$\Rightarrow f_3$ je holomorfna
$f_3 \in \mathcal{H}(D)$

---

(2) Naj bo $f = u + iv$ holomorfna funkcija z danim realnim delom $u(x,y) = x^3 - 3xy^2$. Določi funkcijo f.

CR-sistem: $u_x = 3x^2 - 3y^2 = v_y \to v_1 = 3x^2y - y^3 + C_1(x)$
$u_y = -6xy = -v_x \to v_2 = 3x^2y + C_2(y)$
$\Rightarrow v = 3x^2y - y^3 + C$

$f(x+iy) = x^3 - 3xy^2 + i(3x^2y - y^3 + C) = x^3 + 3x^2yi + 3x(yi)^2 + (yi)^3 + iC = (x+iy)^3 + iC$

Če nismo iznajdljivi: $f(z) = u(x,y) + i\,v(x,y) = \dots$ ($x = \frac{z+\bar z}{2}$, $y = \frac{z-\bar z}{2}$)

TRDITEV: $A \subset D$ množica s stekališčem s v D: $s = \lim_{n\to\infty} a_n$, $a_n \in A \setminus \{s\}$
$f, g \in \mathcal{H}(D)$ in $f|_A = g|_A \Rightarrow f = g$

Nadaljevanje naloge:

$D = \mathbb{C}$; $A = \mathbb{R} \subseteq \mathbb{C}$.
Zadošča določiti $f(x)$ za $x \in \mathbb{R}$: $f(x) = u(x,0) + i v(x,0)$.
Iščemo $v(x,0)$:

$$v_x(x,0) = -u_y(x,0) \Rightarrow v(x,0) = -\int u_y(x,0)\,dx = \int 0\,dx = C$$

$$\Rightarrow f(x) = u(x,0) + i v(x,0) = x^3 + iC \quad \text{za } x \in \mathbb{R} = A$$

$$\overset{z\in\mathbb{C}}{\Longrightarrow} \underbrace{f(z) = z^3 + iC}$$

x zamenjamo z z-jem

---

(4) Določi konstanto k tako, da bo $u(x,y) = e^x(\cos ky + \sin ky)$ realni del neke holomorfne funkcije f in nato določi še funkcijo f.

$$v(x,0) = -\int u_y(x,0)\,dx = -\int [e^x(-k\sin(ky) + k\cos(ky))](x,0)\,dx = -k\int e^x\,dx = -ke^x + C$$

$$\Rightarrow f(\underset{\mathbb{R}}{x}) = u(x,0) + i v(x,0) = e^x - ike^x + iC = e^x(1-ik) + iC$$

$$\Rightarrow f(z) = e^z(1-ik) + iC$$

Preverimo: $u(x,y) = \text{Re}\, f(x+iy)$:

$$f(x+iy) = (1-ik)e^{x+iy} + iC \underset{\text{Euler}}{=} (1-ik)e^x(\cos y + i\sin y) + iC$$

$$\Rightarrow \text{Re}\, f(x+iy) = e^x(\cos y + k\sin y)$$

$\Rightarrow$ Ustreza $k = \pm 1$

$\cos y + k\sin y = \cos ky + \sin ky$
$y = 0$: $1 = 1$
$y = \frac{\pi}{2}$: $0 + k = \cos\frac{k\pi}{2} + \sin\frac{k\pi}{2}$

1. način:

$$C\sin(\delta + y) = C(\sin\delta\cos y + \cos\delta\sin y) \rightarrow \sqrt{1+k^2}\left(\frac{1}{\sqrt{1+k^2}}\cos y + \frac{k}{\sqrt{1+k^2}}\sin y\right) = \sqrt{1+k^2}\sin(y+\delta)$$

Amplituda mora biti enaka $\rightarrow \sqrt{2}\sin(y+\delta) \Rightarrow k = \pm 1$

$1^2 + 1^2 = 2$

2. način:

$u_x = v_y \wedge u_y = -v_x$ če holomorfna, to velja.

$$\Rightarrow u_{xx} = v_{xy} \wedge u_{yy} = -v_{xy} \underset{v\in C}{=} -v_{xy} \Rightarrow u_{xx} + u_{yy} = 0$$

Če $f \in \mathcal{H}(D)$, potem $\Delta u = 0$

Naš primer: $0 = \Delta u = e^x(\cos ky + \sin ky) + e^x(-k^2\cos ky + k^2\sin ky)$

$$\Rightarrow (1-k^2)(\cos ky + \sin ky) = 0$$

$y = 0 \Rightarrow 1 - k^2 = 0$
$\Rightarrow \underbrace{k = \pm 1}$

(5) Določi holomorfno funkcijo z realnim delom $u(x,y) = \frac{1}{2}\log(x^2+y^2)$.

$$x\in\mathbb{R}:\quad v(x,0) = \int v_x(x,0)\,dx = -\int u_y(x,0)\,dx = -\int \frac{1}{2}\Big(\underbrace{\frac{1}{x^2+y^2}2y}_{0}\Big)\Big|_{y=0} dx = C$$

$$f(x) = u(x,0) + i\cdot v(x,0) = \frac{1}{2}\log x^2 + iC$$

Za $x>0$: $f(x) = \log x + iC$

$\Downarrow$ x nadomestimo z $z\in\mathbb{C}$

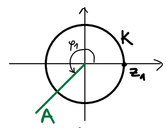
$$f(z) = \log z + iC$$

$$f(z) = \log z = a+ib \iff e^{f(z)} = z$$
$$e^{a+ib} = z$$
$$e^a e^{ib} = z$$
$$e^a(\cos b + i\sin b) = z \iff e^a = |z|,\ b = \arg(z)$$

$\underline{\log z = \log|z| + i\arg(z)}$; obstaja za $z\neq 0$

$\log|z|$: "običajni" logaritem, na $(0,\infty)$

Ali je $\log z$ holomorfna na $\mathbb{C}\setminus\{0\}$?



Za $z_1\in(0,\infty)$ lahko "izberemo" $\arg(z_1)=0$ (argument je določen do večkratnika 2π).
Če se po krožnici K sprehodimo okoli izhodišča, na koncu dohoda zaradi zveznosti dobimo $\arg(z_1) = 2\pi$

$\Rightarrow$ Funkcije $\log z$ ne moremo zvezno definirati na $\mathbb{C}\setminus\{0\}$.

$\log z$ je holomorfen na prerezani kompleksni ravnini $\mathbb{C}\setminus A$, kjer je A nek žarek/poltrak z začetkom v 0.

$$\rightarrow\ \mathbb{C}\setminus A = \{z;\ \underbrace{\arg(z)\in(\varphi_1, \varphi_1+2\pi)}_{\text{odločimo se za interval elementa}}\}$$

Z izbiro $\arg(z)$ definiramo vejo logaritma, ki je holomorfna na $\mathbb{C}\setminus A$.

Če želimo vejo, ki na $(0,\infty)$ sovpada z običajnim logaritmom, lahko prerežemo z $A = (-\infty, 0]$ in izberemo $\arg(z)\in(-\pi,\pi)$. $\rightarrow \arg|_{(0,\infty)} \equiv 0$

---

(6) Določi holomorfno funkcijo f, za katero velja $|f(x+iy)| = (x^2+y^2)e^x$.

Iščemo funkcijo g, da bo realni del vseboval $|f|$:

$$\underbrace{\log f(z)}_{g(z)} = \underbrace{\log|f(z)|}_{u(x,y)} + i\arg(f(z)).$$

$$\rightarrow u(x,y) = \log[(x^2+y^2)e^x]\quad (=\operatorname{Re} g)$$

$$v(x,0) = -\int u_y(x,0)\,dx = -\int \underbrace{\frac{2y}{x^2+y^2}}_{0}\Big|_{y=0} dx = C$$

$$x\in\mathbb{R}: \rightarrow g(x) = u(x,0) + iv(x,0) = \log x^2 + x + iC$$

$x\neq 0$

$$\Rightarrow g(z) = \log z^2 + z + iC = \log f(z)$$

$$\Rightarrow \underline{f(z) = z^2e^{z+iC} = \alpha z^2 e^z;\ |\alpha| = 1}$$

$|f|$ podaja funkcijo do faktorja α, $|\alpha|=1$, natančno

(7) Kje so holomorfne funkcije, podane z naslednjimi potenčnimi vrstami?

a) $f(z) = \sum_{n=0}^{\infty} \binom{2n}{n} z^n$
b) $f(z) = \sum_{n=1}^{\infty} n^p z^n$, kjer je $p \in \mathbb{R}$
c) $f(z) = \sum_{n=0}^{\infty} n! z^n$
d) $f(z) = \sum_{n=0}^{\infty} z^{2^n}$
e) $f(z) = \sum_{n=1}^{\infty} \frac{(z-1-i)^n}{n2^n}$
f) $f(z) = \sum_{n=1}^{\infty} \frac{nz^n}{n!}$
g) $f(z) = \sum_{n=1}^{\infty} 2^n z^{n!}$
h) $f(z) = \sum_{n=1}^{\infty} a_n z^{2n}$ in $g(z) = \sum_{n=1}^{\infty} a_n^2 z^n$, če ima vrsta $\sum_{n=1}^{\infty} a_n z^n$ konvergenčni polmer $R > 0$

Splošno: $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$; $a \in \mathbb{C}$ ... središče vrste
$c_n \in \mathbb{C}$ ... koeficient

$\exists R \in [0, \infty]$ ... konvergenčni polmer $\qquad R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{\sqrt[n]{c_n}}$

- na $\Delta(a,R) = \{z;\ |z-a| < R\}$ vrsta konvergira
- za $|z-a| > R$ vrsta divergira
- če je $R > 0$, je $f \in \mathcal{H}(\Delta(a,R))$

(a) $f(z) = \sum_{n=0}^{\infty} \binom{2n}{n} z^n \rightarrow a = 0$

$$R = \lim_{n\to\infty} \frac{\binom{2n}{n}}{\binom{2n+2}{n+1}} = \lim_{n\to\infty} \frac{(2n)!\ (n+1)!\ (n+1)!}{n!\, n!\ (2n+2)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}$$

$\Rightarrow f(z) \in \mathcal{H}(\Delta(0, \frac{1}{4}))$

(b) $f(z) = \sum_{n=1}^{\infty} n^p z^n$; $p \in \mathbb{R} \rightarrow a = 0$

$$R = \lim_{n\to\infty} \left|\frac{n^p}{(n+1)^p}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1$$

$\Rightarrow f(z) \in \mathcal{H}(\Delta(0,1))$

(c) $f(z) = \sum_{n=0}^{\infty} n!\, z^n \rightarrow a = 0$

$$R = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$\Rightarrow f(z)$ ni holomorfna funkcija

(d) $f(z) = \sum_{n=0}^{\infty} z^{2^n} \rightarrow a = 0$

$c_m = \begin{cases} 1 ; & m = 2^n \\ 0, & \text{sicer} \end{cases} \quad \rightarrow \frac{1}{R} = \limsup_{m\to\infty} \sqrt[m]{|c_m|} = 1$

koeficient pri z^m $\qquad \sqrt[m]{|c_m|} = \begin{cases} 1 ; & m = 2^n \\ 0 ; & \text{sicer} \end{cases}$

$\rightarrow R = 1$

$\Rightarrow f(z) \in \mathcal{H}(\Delta(0,1))$

Univerzalno: $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$

(e) $f(z) = \sum_{n=1}^{\infty} \frac{(z-1-i)^n}{n2^n} \rightarrow a = 1+i$

$$R = \lim_{n\to\infty} \left|\frac{n\cdot 2^n}{(n+1)2^{n+1}}\right|^{-1} = \lim_{n\to\infty} \left|\frac{n}{2n+2}\right|^{-1} = 2 \quad \Rightarrow f(z) \in \mathcal{H}(\Delta(1+i, 2))$$

(g) $f(z)=\sum_{n=1}^{\infty} 2^n z^{n!} \to a=0$; $c_m=\begin{cases} 2^m ; & m=n! \\ 0 ; & \text{sicer} \end{cases}$

$\frac{1}{R}=\limsup_{m\to\infty} \sqrt[m]{|c_m|} = \limsup_{n\to\infty} \sqrt[n!]{2^n} = \limsup_{n\to\infty} 2^{\frac{n}{n!}} = 1 \to R=1$

$\Rightarrow \underline{f(z)\in\mathcal{H}(\Delta(0,1))}$

(f) $f(z)=\sum_{n=1}^{\infty} \frac{nz^n}{n!} \to a=0$

$R=\lim_{n\to\infty} \frac{n!}{(n-1)!} = \infty$

$\Rightarrow \underline{f(z)\in\mathcal{H}(\mathbb{C})}$

Dodatek: $f(z)=\sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^n$

$e^z=\sum_{m=0}^{\infty} \frac{1}{m!} z^m \Rightarrow \underline{f(z)= z\cdot\sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = z\cdot e^z}$

(h) $f(z)=\sum_{n=1}^{\infty} a_n z^{2n}$
$g(z)=\sum_{n=1}^{\infty} a_n^2 z^n$

$h(z)=\sum_{n=1}^{\infty} a_n z^n$ ima $R>0 \to \frac{1}{R}=\limsup_{n\to\infty} \sqrt[n]{|a_n|}$

$f(z)$: $a=0$
$c_m=\begin{cases} a_m , & m=2n \\ 0 ; & \text{sicer} \end{cases}$

$\frac{1}{R_f}=\limsup_{n\to\infty} \sqrt[2n]{|a_n|} = \limsup_{n\to\infty} \sqrt{\sqrt[n]{|a_n|}} = \sqrt{\frac{1}{R}} \to R_f=\sqrt{R}$

$\Rightarrow \underline{f(z)\in\mathcal{H}(\Delta(0,\sqrt{R}))}$

2. način:
$h\in\mathcal{H}(\Delta(0,R))$: $\exists h(z)$, če $|z|<R$
Opazimo: $f(z)=h(z^2)$
$\Rightarrow \exists f(z)$, če $|z^2|<R \Leftrightarrow |z|<\sqrt{R}$

$g(z)$: $a=0$, $c_n=a_n^2$

$\frac{1}{R_g}=\limsup_{n\to\infty} \sqrt[n]{a_n^2} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}^2 = \frac{1}{R^2} \to R_g=R^2$

$\Rightarrow \underline{g(z)\in\mathcal{H}(\Delta(0,R^2))}$

---

(8) Funkcijo $f(z)=\sin z$ razvij okrog točke $z_0=i$.

Če je $f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n$, potem je $c_n=\frac{f^{(n)}(a)}{n!}$.

$\sin z=\sum_{n=0}^{\infty} c_n(z-i)^n$
$\to w=z-a=z-i$

$\Rightarrow \sin z = \sin(w+i) =$
$=\sin w\cdot\cos i + \sin i\cdot\cos w =$
$=\left[\sum_{n=0}^{\infty} (-1)^n \frac{w^{2n+1}}{(2n+1)!}\right]\cos i + \left[\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!}\right]\sin i =$
$\underline{= \cos i\sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^{2n+1}}{(2n+1)!} + \sin i\sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^{2n}}{(2n)!}}$

Dodatek:

$z = x + iy$

$e^{iz} = e^{-y}(\cos x + i\sin x)$

$e^{-iz} = e^{y}(\cos x - i\sin x)$

Za $\varphi \in \mathbb{R}$: $\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2}$ in $\sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i}$

$\overset{z\in\mathbb{C}}{\Longrightarrow}$ $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ in $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

Vemo: $\cosh t = \frac{e^{t}+e^{-t}}{2}$ in $\sinh t = \frac{e^{t}-e^{-t}}{2}$

$\Rightarrow$ $\cos z = \cosh(iz)$ in $\sin z = -i\sinh(iz)$

$\sin(x+iy) = \sin x\cos(iy) + \sin(iy)\cos x = \sin x \cdot \cosh(-y) - i\cos x \cdot \sinh(-y)$

$\rightarrow$ $\underline{\sin(x+iy) = \sin x \cdot \cosh y + i\cos x \cdot \sinh y}$

$\Rightarrow$ $\sin i = i \cdot \sinh 1 = i\,\frac{e-e^{-1}}{2}$

$\cos(x+iy) = \cos x\cos(iy) - \sin x\sin(iy) = \cos x\cosh(-y) + i\sin x\sinh(-y)$

$\rightarrow$ $\underline{\cos(x+iy) = \cos x\cosh y - i\sin x\sinh y}$

$\Rightarrow$ $\cos i = \cosh 1 = \frac{e+e^{-1}}{2}$

PARAMETRIZACIJA:

$z = z(t)\ ;\ t\in[a,b]$

$z(t) = u(t) + i v(t)$ $\quad\rightarrow\quad dz = du + i\,dv = \dot u(t)\,dt + i\dot v(t)\,dt$

$f(z) = u_1 + iv_1$

$$\int_K f(z)\,dz = \int_a^b \Big[\underbrace{(u_1\dot u - v_1\dot v)}_{U} + i\underbrace{(v_1\dot u + u_1\dot v)}_{V}\Big]\,dt = \int_a^b U\,dt + i\int_a^b V\,dt$$

---

(9) Izračunaj integral kompleksne funkcije

$$\int_{\partial D} f(z)dz,$$

kjer je

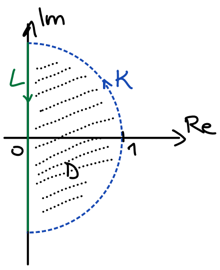
a) $f(z) = |z|$ in $D = \{z \mid |z| < 1,\ \mathrm{Re}\, z > 0\}$.

b) $f(z) = \frac{z}{\bar z}$ in $D = \{z \mid 1 < |z| < 2,\ \mathrm{Re}\, z > 0\}$.

c) $f(z) = \bar z$ in D območje s kosoma gladkim robom.

d) f holomorfna funkcija v okolici $\overline{D}$ in D območje s kosoma gladkim robom.

(a) $f(z) = |z|$; $D = \{z\in\mathbb{C} \mid |z|<1, \mathrm{Re}\,z > 0\}$



$\partial D = K \cup L$: $\quad I = \int_{\partial D} f(z)dz = \underbrace{\int_K f(z)dz}_{=I_K} + \underbrace{\int_L f(z)dz}_{=I_L}$

$\left.\begin{array}{l} z = \cos\varphi + i\sin\varphi = e^{i\varphi} \\ dz = ie^{i\varphi}\,d\varphi \end{array}\right\}$ parametrizacija po K

$$\underline{I_K} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overbrace{|e^{i\varphi}|}^{=1}\, i e^{i\varphi}\,d\varphi = i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\varphi}d\varphi = i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\,d\varphi + i^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\varphi\,d\varphi = i\sin\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \cos\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$= i(1+1) = \underline{2i}$

Lahko bi integrirali "kot običajno":

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\varphi} = \frac{1}{i}\, e^{i\varphi}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -i\left(e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}\right) = -i(i+i) = 2$$

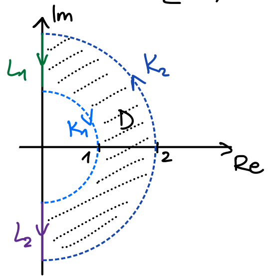
teče "nazaj"

$\left.\begin{array}{l} z = iy\ ;\ y\in[-1,1] \\ dz = i\,dy \end{array}\right\}$ parametrizacija po L

$$\underline{I_L} = \int_1^{-1} |iy|\, i\,dy = i\int_1^{-1}|y|\,dy = i\cdot 2\cdot\int_1^0 y\,dy = i\cdot 2\cdot\frac{-1}{2} = \underline{-i}$$

$\Rightarrow$ $\underline{I} = I_K + I_L = \underline{i}$

(b) $f(z)=\dfrac{z}{\bar z}$; $D=\{z\in\mathbb{C} \mid 1<|z|<2,\ \operatorname{Re} z>0\}$



parametrizacija krožnice s polmerom R: $z=Re^{i\varphi}$
$dz=Rie^{i\varphi}\,d\varphi$

$$\int_{K_R^+} f(z)dz=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{Re^{i\varphi}}{Re^{-i\varphi}}\,Rie^{i\varphi}d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Rie^{3i\varphi}\,d\varphi=\frac{Ri}{3i}\,e^{3i\varphi}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=$$

desna polravnina

$$=\frac{R}{3}\left(e^{i\frac{3\pi}{2}}-e^{-i\frac{3\pi}{2}}\right)=\frac{R}{3}(-i-i)=-\frac{2Ri}{3}$$

nasprotna orientacija

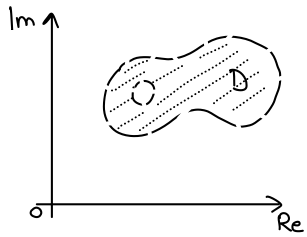
$I_{K_1}=+\dfrac{2i}{3}$ in $I_{K_2}=-\dfrac{4i}{3}$

$z=iy$; $y\in\overleftarrow{[-2,-1]}\cup\overleftarrow{[1,2]}=P$
$dz=i\,dy$ } parametrizacija po $L_1\cup L_2$

$$I_{L_1\cup L_2}=\int_P\frac{iy}{-iy}\,i\,dy=-i\int_P dy=2i$$

$$\Rightarrow I=I_{K_1}+I_{K_2}+I_{L_1\cup L_2}=\frac{2i}{3}-\frac{4i}{3}+2i=\frac{4}{3}i$$

(c) $f(z)=\bar z$; D območje s kosoma gladkim robom



$z=x+iy$
$dz=dx+i\,dy$

$$\int_{\partial D}\bar z\,dz=\int_P(x-iy)(dx+idy)=\int_P x\,dx+y\,dy+i\int_P -y\,dx+x\,dy=$$

območje parametra

$$=\int_{\partial D}x\,dx+y\,dy+i\int_{\partial D}-y\,dx+x\,dy\overset{\text{Green}}{=}\int_D(0-0)\,dx\,dy+i\int_D(1+1)\,dx\,dy=2i\int_D dx\,dy=2i\cdot P(D)$$

(d) D območje s kosoma gladkim robom
f holomorfna funkcija v okolici $\bar D$

$$I=\int_{\partial D}f(z)dz=\int_{\partial D}[u(x,y)+iv(x,y)](dx+idy)=\int_{\partial D}u\,dx-v\,dy+i\int_{\partial D}u\,dy+v\,dx=$$

$f=u+iv$
$dz=dx+idy$

$$\overset{\text{Green}}{=}\int_D\underbrace{(-v_x-u_y)}_{0\ \Leftarrow\ u_y=-v_x}dx\,dy+i\int_D\underbrace{(u_x-v_y)}_{0\ \Leftarrow\ u_x=v_y}dx\,dy=0$$

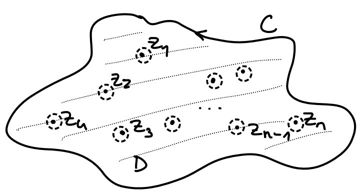
---

(11) Naj bo $n>1$ in $z_1,z_2,\ldots,z_n$ točke v kompleksni ravnini. Izračunaj integral

$$I=\int_C\frac{dz}{(z-z_1)(z-z_2)\cdots(z-z_n)},$$

a) če so vse točke $z_1,z_2,\ldots,z_n$ različne.
b) če so vse točke $z_1,z_2,\ldots,z_n$ poljubne.

C krivulja, ki enkrat objame $z_1,\ldots,z_n$



$C=\partial D$ $\qquad f(z)=\dfrac{1}{(z-z_1)\cdots(z-z_n)}$

$\rightarrow I=\displaystyle\int_{\partial D}f(z)dz$

Iz D izrežemo $\Delta(z_i,\varepsilon)=\Delta_i$: $\bar\Delta_i\cap\bar\Delta_j=\emptyset$; $i\neq j$ $\Rightarrow D'=D\setminus\bigcup_{1}^{n}\Delta_i$
$\bar\Delta_i\subseteq D$

Cauchyjev izrek: $\int_{\partial D'} f(z)\,dz = 0$

$$\to \int_{\partial D} f(z)\,dz = \int_{\partial D'} f(z)\,dz + \sum_{1}^{n} \int_{\partial \Delta_i} f(z)\,dz = \sum_{1}^{n} \int_{\partial \Delta_i} f(z)\,dz =$$

Vemo: $\displaystyle\int_{\underbrace{|z-z_0|=r}_{\partial\Delta_0}} \frac{g(z)}{z-z_0}\,dz = 2\pi i\, g(z_0)$; $g \in \mathcal{H}(\underbrace{\overline{\Delta}(z_0,r)}_{\overline{\Delta}_0})$

(a) $\displaystyle\int_{\partial\Delta_i} f(z)\,dz = \int_{\partial\Delta_i} \frac{g(z)}{z-z_i}\,dz = 2\pi i \cdot g(z_i)$; $\displaystyle g(z) = \frac{1}{(z-z_1)\cdots(z-z_{i-1})(z-z_{i+1})\cdots(z-z_n)} \in \mathcal{H}(\dots)$ $z_i \neq z_j,\ i \neq j$

$$\int_{\partial D} f(z)\,dz = \sum_{k=1}^{n} 2\pi i\, g(z_k) = 2\pi i \cdot \sum_{k=1}^{n} \frac{1}{(z_k-z_1)\cdots(z_k-z_{k-1})(z_k-z_{k+1})\cdots(z_k-z_n)} = 0$$ ne vemo kako

2. način: $\displaystyle f(z) \underset{\text{parcialni ulomki}}{=} = \frac{a_1}{z-z_1} + \frac{a_2}{z-z_2} + \dots + \frac{a_n}{z-z_n}$

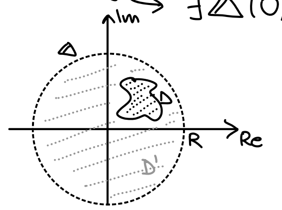
$$\to \int_{\Delta_k} f(z)\,dz = \sum_{\ell=1}^{n} \underbrace{\int_{\partial\Delta_\ell} \frac{a_\ell}{z-z_\ell}\,dz}_{I_\ell = \begin{cases} 0; & \ell\neq k \\ 2\pi i a_\ell; & \ell = k\end{cases}} = \sum_{\ell=1}^{n} 2\pi i a_\ell = 2\pi i \sum_{1}^{n} a_\ell \underset{(*)}{=} 0$$

$$f(z) = \frac{a_1\overbrace{(z-z_2)\cdots(z-z_n)}^{z^{n-1}+\dots} + \dots + a_n\overbrace{(z-z_1)\cdots(z-z_{n-1})}^{z^{n-1}+\dots}}{(z-z_1)(z-z_2)\cdots(z-z_n)} = \frac{z^{n-1}(a_1+\dots+a_n)+\dots}{(z-z_1)\cdots(z-z_n)} = \frac{1}{(z-z_1)\cdots(z-z_n)}$$

$\Rightarrow a_1 + \dots + a_n = 0 \quad (*) \qquad \Rightarrow \displaystyle\int_{\partial D} f(z)\,dz = 0$

(b) D omejeno območje

$\hookrightarrow \exists \Delta(0,R):\ \overline{D} \subseteq \Delta(0,R)$



$D' = \Delta \setminus \overline{D}$

$\displaystyle\int_{\partial D'} f(z)\,dz \overset{\text{Cauchy}}{=} 0 \qquad 0 = \int_{\partial D'} = \int_{\partial\Delta} - \int_{\partial D} \quad \Rightarrow \int_{\partial D} = \int_{\partial \Delta}$

$\displaystyle\int_{\partial D} f(z)\,dz = \int_{\partial\Delta} f(z)\,dz$ za vsak dovolj velik R

$$\int_{\partial D} f(z)\,dz = \lim_{R\to\infty} \int_{|z|=R} f(z)\,dz = 0$$

$$\left|\int_{|z|=R} f(z)\,dz\right| \underset{z=Re^{i\varphi}}{=} \left|\int_0^{2\pi} f(Re^{i\varphi})\,Ri e^{i\varphi}\,d\varphi\right| \le \int_0^{2\pi} |f(Re^{i\varphi})|\,R\,d\varphi \underset{|ie^{i\varphi}|=1}{=} \int_0^{2\pi} \left|\frac{R}{(Re^{i\varphi}-z_1)\cdots(Re^{i\varphi}-z_n)}\right| d\varphi \le$$

$|Re^{i\varphi} - z_k| \geq |Re^{i\varphi}| - |z_k|$, $|Re^{i\varphi}| = R$

$$\le \int_0^{2\pi} \underbrace{\frac{R}{(R-|z_1|)\cdots(R-|z_n|)}}_{\substack{\text{neodvisno od } \varphi \\ g(R)}}\,d\varphi = 2\pi g(R)$$

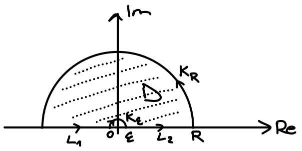
$\displaystyle\lim_{R\to\infty} g(R) = \lim_{R\to\infty} \frac{R}{R^2+\dots} = 0 \qquad \to \qquad 0 \le \left|\int_{|z|=R} f(z)\,dz\right| \le g(R)$, $\quad g(R) \xrightarrow{R\to\infty} 0$, $\quad \left|\int_{|z|=R} f(z)\,dz\right| \to 0$

$$\Rightarrow \int_{\partial D} f(z)\,dz = 0$$

(21) S pomočjo kompleksne integracije izračunaj integral

$$\int_0^\infty \frac{\sin x\,dx}{x}.$$

Izračunaj s pomočjo $f(z)=\frac{e^{iz}}{z}$ in $D=\{z\in\mathbb{C};\ \varepsilon<|z|<R \wedge \operatorname{Im} z>0\}$



$$\int_{\partial D} f(z)\,dz = 0, \text{ ker je } f \text{ holomorfna na } D$$

$$\int_{\partial D} = \int_{K_1} + \int_{K_2} + \int_{L_1} + \int_{L_2}$$

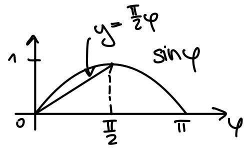
K_R: $z=Re^{i\varphi}$ $\rightarrow$ $\displaystyle\int_{K_R} f(z)\,dz = \int_0^\pi \frac{e^{iRe^{i\varphi}}}{Re^{i\varphi}}\, Rie^{i\varphi}\,d\varphi$

$$\lim_{R\to\infty}\int_{K_r}\ldots = 0$$

$$\left|\int_{K_R}\ldots\right| = \left|\int_0^\pi e^{iR\cos\varphi - R\sin\varphi}\,d\varphi\right| \le \int_0^\pi \underbrace{|e^{iR\cos\varphi - R\sin\varphi}|}_{= e^{-R\sin\varphi}\in\mathbb{R}}\,d\varphi \le \int_0^\pi e^{-R\sin\varphi}\,d\varphi =$$

$$= 2\int_0^{\frac{\pi}{2}} e^{-R\sin\varphi}\,d\varphi \le 2\int_0^{\frac{\pi}{2}} e^{-R\frac{2\varphi}{\pi}}\,d\varphi = -\frac{\pi}{R}\left(e^{-R}-1\right) \xrightarrow{R\to\infty} 0$$

$\frac{\pi}{R} \xrightarrow{R\to\infty} 0$, $e^{-R} \xrightarrow{R\to\infty} 0$



za $\varphi\in[0,\frac{\pi}{2}]$ je $\sin\varphi \ge \frac{2\varphi}{\pi}$

$$\rightarrow \lim_{R\to\infty}\int_{K_R} = 0$$

$L_1\cup L_2$: ($z=x$, $dz=dx$)

$$\int_{\underbrace{[-R,-\varepsilon]\cup[\varepsilon,R]}_{L}} f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{\cos x + i\sin x}{x}\,dx + \int_\varepsilon^R \frac{\cos x + i\sin x}{x}\,dx = \underbrace{\int_L \frac{\cos x}{x}\,dx}_{\lim = 0 \ *} + i\int_L \frac{\sin x}{x}\,dx = \cdots = 2i\int_\varepsilon^\infty \frac{\sin x}{x}\,dx$$

$$\downarrow$$

$$2i\int_0^\infty \frac{\sin x}{x}\,dx = 2iI$$

$*$ $\displaystyle\int_{A\cup(-A)} g(x)\,dx = 0$ za liho g in $\displaystyle\int_{A\cup(-A)} g(x)\,dx = 2\int_A g(x)\,dz$ za sodo g

$z=\varepsilon e^{i\varphi}$
$dz=\varepsilon i e^{i\varphi}$

enako kot na K_R

$$\int_{K_\varepsilon} f(z)\,dz = \int_\pi^0 i\underbrace{e^{i\varepsilon e^{i\varphi}}}_{\xrightarrow{\varepsilon\to 0} e^0 = 1}\,d\varphi \xrightarrow{\varepsilon\to 0^+} \int_\pi^0 i\,d\varphi = -\pi i$$

integral s parametrom: $\displaystyle\int_a^b g(x,\varepsilon)\,dx \xrightarrow[\uparrow\ g \text{ zvezna}]{\varepsilon\to\varepsilon_1} \int_a^b g(x,\varepsilon_1)\,dx$

SKUPNO: $\displaystyle\int_{\partial D} f(z)\,dz = \int_{K_R} f(z)\,dz + \int_L f(z)\,dz + \int_{K_\varepsilon} f(z)\,dz \xrightarrow[\varepsilon\to 0]{R\to\infty} 0 = 0 + 2iI - \pi i \implies \underline{I = \frac{\pi}{2}}$

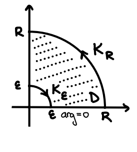
---

(23) Naj bo $a>0$, $0<p<1$ in $0<\varepsilon<R$. S pomočjo kompleksne integracije po robu območja $D=\{z \mid |z|\in(\varepsilon,R),\ \operatorname{Re} z, \operatorname{Im} z>0\}$ izračunaj integrala

$$I_1 = \int_0^\infty x^{p-1}\cos ax\,dx \text{ in } I_2 = \int_0^\infty x^{p-1}\sin ax\,dx.$$

$f(z) = z^{p-1}e^{iaz}$: $\displaystyle\int_{\partial D} f(z)\,dz = 0$ $\leftarrow f\in\mathcal{H}(\bar{D})$

$x\in\mathbb{R}$: $e^{iax} = \cos ax + i\sin ax$



KOMPLEKSNA POTENCA: $z^{\alpha} = e^{\alpha \log z}$; $\alpha \in \mathbb{C}$

$\log z = \log|z| + i \arg(z)$

Izberemo vejo logaritma, da bo z^{p-1} holomorfna na okolici $\bar{D}$ in ima za $z=x$, $x>0$, potenca $z^{p-1} = x^{p-1}$ običajen pomen.

$\log z$ definiramo na prerezani ravnini: $\mathbb{C} \setminus (-\infty, 0]$ in $\arg z \in (-\pi, \pi)$

Na ta način je z^{p-1} holomorfna funkcija $\Rightarrow$ $f(z)$ je holomorfna funkcija

$$\int_{\partial D} = \int_{K_R} + \int_{iR}^{i\varepsilon} + \int_{K_\varepsilon} + \int_{\varepsilon}^{R}$$

integral po daljici

$z = Re^{i\varphi}$
$dz = iRe^{i\varphi}$

$z^{p-1} = (Re^{i\varphi})^{p-1} = R^{p-1} e^{(p-1)i\varphi}$ Ali velja = ? *

$$(1) \quad \int_{K_R} z^{p-1} e^{iaz} dz = \left| \int_{K_R} \dots \right| = \left| \int_0^{\frac{\pi}{2}} R^{p-1} e^{(p-1)i\varphi} e^{iaRe^{i\varphi}} \cdot iRe^{i\varphi} d\varphi \right| = \qquad |e^{it}| = 1, \ t \in \mathbb{R}$$

$$= \left| \int_0^{\frac{\pi}{2}} R^{p-1} e^{i(aRe^{i\varphi} + (p-1)\varphi)} \cdot iRe^{i\varphi} d\varphi \right| \leq \int_0^{\frac{\pi}{2}} |\dots| d\varphi = \int_0^{\frac{\pi}{2}} R^p e^{-aR\sin\varphi} d\varphi \leq$$

$\varphi \in [0, \frac{\pi}{2}] \rightarrow \sin\varphi \geq \frac{2\varphi}{\pi}$

$$\leq \int_0^{\frac{\pi}{2}} R^p e^{-a \frac{2\varphi}{\pi} R} d\varphi = R^p \left(-\frac{\pi}{a2R} e^{-a\frac{2\varphi}{\pi}R} \right) \Big|_0^{\frac{\pi}{2}} =$$

$$= \underbrace{-\frac{R^{p-1}\pi}{2a}}_{\downarrow R\to\infty \ 0} \Big(\underbrace{e^{-aR} - 1}_{\downarrow R\to\infty \ 1} \Big) \xrightarrow{R\to\infty} \underline{0}$$

$$* \quad \log z = \log|z| + i\arg(z) = \\ = \log|Re^{i\varphi}| + i\arg(Re^{i\varphi}) = \\ = \log|Re^{i\varphi}| + i\cdot\varphi = \qquad \text{za } \varphi \in [0, -\tfrac{\pi}{2}] \\ = \log R + i\varphi$$

$$z^{p-1} = e^{(p-1)\log z} = e^{(p-1)(\log R + i\varphi)} = \\ = e^{(p-1)\log R} \cdot e^{(p-1)i\varphi} = \\ = R^{p-1} \cdot e^{(p-1)i\varphi}$$

$$(2) \quad \int_{iR}^{i\varepsilon} f(z)dz = \int_R^{\varepsilon} (iy)^{p-1} e^{iaiy} \cdot i dy = \int_R^{\varepsilon} y^{p-1} e^{i\frac{\pi}{2}(p-1)} e^{-ay} i dy = i \cdot e^{i\frac{\pi}{2}(p-1)} \underbrace{\int_R^{\varepsilon} y^{p-1} e^{-ay} dy}_{\int_{\infty}^{0} y^{p-1} e^{-ay} dy}$$

$z = iy$
$dz = idy$

$$(iy)^{p-1} = e^{(p-1)\log(iy)} = e^{(p-1)(\log y + \frac{\pi}{2}i)} = \\ = y^{p-1} \cdot e^{i\frac{\pi}{2}(p-1)}$$

$t = ay$
$dt = a dy$

$$\xrightarrow[\varepsilon\to 0]{R\to\infty} -i\, e^{i\frac{\pi}{2}(p-1)} \int_0^{\infty} a^{-p} t^{p-1} e^{-t} dt =$$

Gamma funkcija: $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$

$$= \underline{-i\, e^{i\frac{\pi}{2}(p-1)} a^{-p} \Gamma(p)}$$

$$(3) \quad \int_{K_\varepsilon} f(z)dz = \int_{\frac{\pi}{2}}^{0} \underbrace{\varepsilon^{p-1} \underbrace{e^{(p-1)i\varphi} \cdot e^{ia\varepsilon e^{i\varphi}}}_{} \cdot i\varepsilon e^{i\varphi}}_{\varepsilon^p \ \downarrow \ 0} d\varphi \rightarrow \underline{0}$$

$z = \varepsilon e^{i\varphi}$
$dz = i\varepsilon e^{i\varphi}$

$g(\varepsilon, p)$ zvezna, ker $p > 0$

$$(4) \quad \int_{\varepsilon}^{R} f(z)dz = \int_{\varepsilon}^{R} x^{p-1} e^{iax} dx = \int_{\varepsilon}^{R} x^{p-1} (\cos ax + i\sin ax) dx \xrightarrow[R\to\infty]{\varepsilon\to 0} \int_0^{\infty} x^{p-1}\cos ax\, dx + i\int_0^{\infty} x^{p-1} \sin ax\, dx$$

$z = x$
$dz = dx$

$$\Rightarrow \int_{\varepsilon}^{R} f(z)dz = I_1 + iI_2$$

$$0 = \int_{K_R} \dots + \int_{iR}^{i\varepsilon} \dots + \int_{R_\varepsilon} \dots + \int_{\varepsilon}^{R} \dots = 0 - a^{-p}\Gamma(p)\, i\, e^{i\frac{\pi}{2}(p-1)} + 0 + I_1 + iI_2$$

$$I_1 + iI_2 = a^{-p}\Gamma(p)\, i\, e^{i\frac{\pi}{2}(p-1)} = a^{-p}\Gamma(p)\, i\left(\cos\left[\tfrac{\pi}{2}(p-1)\right] + i\sin\left[\tfrac{\pi}{2}(p-1)\right]\right)$$

$$\Rightarrow \underbrace{I_1 = a^{-p}\Gamma(p)\cos\frac{p\pi}{2}} \quad \text{in} \quad \underbrace{I_2 = a^{-p}\Gamma(p)\sin\frac{p\pi}{2}}$$

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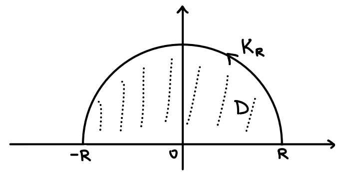
(20) S pomočjo kompleksne integracije izračunaj integral

$$\int_0^\infty \frac{\cos x\, dx}{1+x^2}.$$



$$f(z) = \frac{e^{iz}}{1+z^2} = \frac{e^{iz}}{(z-i)(z+i)} = \frac{g(z)}{z-i}$$

$$\int_{\partial D} f(z)dz \overset{\text{vstavimo } z=i}{=} 2\pi i\, \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

$$\int_{\partial D} \dots = \int_{K_R} \dots + \int_{-R}^{R}$$

$z = Re^{i\varphi}$

$$\int_0^\pi \frac{e^{iRe^{i\varphi}} Rie^{i\varphi}}{1+R^2e^{2i\pi}}\, d\varphi = I_1 \qquad |I_1| \le \int_0^\pi \left| \frac{Re^{iR(\cos\varphi + i\sin\varphi)}}{1+R^2e^{Ri\varphi}} \right| d\varphi = \int_0^\pi \left| \frac{Re^{-R\sin\varphi}}{1+R^2e^{Ri\varphi}} \right| d\varphi \le$$

$$\Rightarrow I_1 = 0 \qquad \le \int_0^\pi \left| \frac{Re^{-R\sin\varphi}}{R^2-1} \right| d\varphi \le \frac{R\pi}{R^2-1} \xrightarrow{R\to\infty} 0$$

$$\underbrace{I_2 = \int_{-R}^{R} \frac{e^{ix}}{1+x^2}dx} = \int_{-R}^{R} \frac{\cos x + i\sin x}{1+x^2}dx = \underbrace{\int_{-R}^{R} \frac{\cos x}{1+x^2}dx}_{\text{soda funkcija}} + \underbrace{i\int_{-R}^{R} \frac{\sin x}{1+x^2}dx}_{0 \text{ (liha funkcija)}} = 2\int_0^R \frac{\cos x}{1+x^2}dx \xrightarrow{R\to\infty} \underbrace{2I}$$

$$\Rightarrow \frac{\pi}{e} = 2I \quad \Rightarrow \underbrace{I = \frac{\pi}{2e}}$$

---

(24) Ali obstaja taka funkcija f, holomorfna v okolici 0, da za dovolj velika naravna števila n velja

a) $f\left(\frac{1}{n}\right) = \sin\left(\frac{n\pi}{2}\right)$?

b) $f\left(\frac{1}{n}\right) = \frac{1}{2n+1}$?

(a) $\sin\frac{n\pi}{2}$ zavzema vrednosti $-1, 0, 1$

f holomorfna v $0 \Rightarrow f$ zvezna v 0

$f(\frac{1}{n})$ nima limite $n\to\infty \Rightarrow$ ne obstaja $\lim\limits_{z\to 0} f(z) \qquad \Rightarrow \underbrace{\text{ne obstaja}}$

(b) $f(0) = \lim\limits_{n\to\infty} f(\frac{1}{n}) = 0$

f poznamo na $A = \{\frac{1}{n}, n \ge n_0\}$, množica s stekališčem

$f(z) = ?$

poznamo za $z \in A$: $f(z) = \dfrac{1}{2\cdot\frac{1}{z}+1} = \dfrac{z}{z+2}$

$g(z) = \frac{z}{z+2}$ holomorfna v okolici 0

$\underbrace{f = g}$ (ustreza zahtevi)

Dodatno: Če je U povezana okolica 0, $f \in \mathcal{H}(U)$, potem $f = g$ edina možnost.
(f in g sovpadata na A /s stekališčem/).

(25) Ali obstaja taka funkcija f, holomorfna v okolici 0, da za dovolj velika naravna števila n velja

$$n^{-\frac{5}{2}} < \left|f\left(\frac{1}{n}\right)\right| < 3n^{-\frac{5}{2}}?$$

$z = \frac{1}{n} \rightarrow n = \frac{1}{z}$

$z^{\frac{5}{2}} < |f(z)| < 3z^{\frac{5}{2}}$

$z = x > 0: \quad f(x) = 2x^{\frac{5}{2}}$

$f \in \mathcal{O}$ (okolica 0) $\Rightarrow f(z) = \sum_{k=0}^{\infty} a_n z^n$

$$\underset{\substack{\downarrow n\to\infty \\ 0}}{n^{-\frac{5}{2}}} < \underbrace{|a_0 + a_1 \tfrac{1}{n} + \cdots|}_{\substack{\downarrow \\ |a_1|}} < \underset{\substack{\downarrow n\to\infty \\ 0}}{3n^{-\frac{5}{2}}} \quad /\cdot n \qquad \Rightarrow a_1 = 0$$

$n^{-\frac{1}{2}} < |a_2 + a_3 \frac{1}{n} + \cdots| < 3n^{-\frac{1}{2}} \rightarrow a_2 = 0$

$n^{\frac{1}{2}} < |a_3 + a_4 \frac{1}{n} + \cdots| < 3n^{\frac{1}{2}} \rightarrow a_3 = 0$

$\times$

$\Rightarrow$ $\underline{f \text{ ne more biti holomorfna}}$

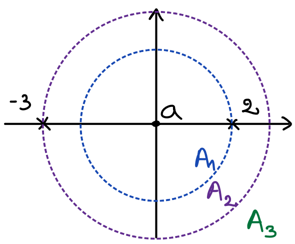
---

(13) Določi vse možne Laurentove razvoje s središčem v $a = 0$ za funkcijo f, podano s predpisom

$$f(z) = \frac{2z+1}{z^2+z-6}.$$

$f \in \mathcal{H}(\overbrace{A(a; r, R)}^{D}) \Rightarrow \forall z \in D: f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$

$A(a;r,R) = \{z \mid r < |z-a| < R\}$



$A_1 = A(0;\ 0, 2)$

$A_2 = A(0;\ 2, 3)$

$A_3 = A(0;\ 3, \infty)$

- Razvoj na A_1:

$f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$

$\frac{A}{z+3} + \frac{B}{z-2} = \frac{Az - 2A + Bz + 3B}{(z+3)(z-2)} \rightarrow A + B = 2 \wedge 3B - 2A = 1 \Rightarrow A = B = 1$

$\Rightarrow f(z) = \frac{1}{z+3} + \frac{1}{z-2}$

$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n; \ |q| < 1$

$\Rightarrow \frac{1}{z+3} = \frac{1}{3(\frac{z}{3}+1)} = \frac{1}{3}\frac{1}{1+\frac{z}{3}} \overset{|z|<3}{=} \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z}{3}\right)^n = \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^{n+1}}}_{} +$

$\Rightarrow \frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} \underset{|z|<2}{=} -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \underbrace{\sum_{n=0}^{\infty}\frac{-z^n}{2^{n+1}}}_{} +$

- Razvoj na A_2: $\frac{1}{z+3} = \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^{n+1}}}_{} +$

enako kot pri A_1

$|\frac{2}{z}| < 1 \rightarrow |z| > 2$ $\qquad m = -(n+1)$

$\frac{1}{z-2} = \underbrace{\frac{1}{z}\frac{1}{1-\frac{2}{z}}}_{\substack{\text{na ta način} \\ \text{dobimo razvoj} \\ \text{izven kroga}}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} = \underbrace{\sum_{m=-\infty}^{-1} 2^{-1-m} z^m}_{\text{glavni del}} +$

$+$

- Razvoj na A_3:

$\frac{1}{z+3} = \frac{1}{z(1+\frac{3}{z})} \overset{|z|>3}{=} \sum_{n=0}^{\infty}\frac{(-3)^n}{z^{n+1}} = \underbrace{\sum_{m=-\infty}^{-1}(-3)^{-m-1} z^m}_{\text{glavni del}} +$

OPOMBA: Če bi imeli višje stopnje; npr. $\frac{A}{(z+7)^3} = A(z+7)^{-3} = A(z(1+\frac{7}{z}))^{-3} = A(7(1+\frac{z}{7}))^{-3}$

$$\underset{\uparrow \atop |t|<1}{(1+t)^{\alpha}} = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n$$

---

a izolirana singularnost za $f \in \mathcal{H}(A(a;0,r))$, $r>0$:

1) a je odpravljiva, če je glavni del enak 0

2) a je pol stopnje $n \in \mathbb{N}$, če glavni del enak $\frac{\overset{\neq 0}{c_{-n}}}{(z-a)^n} + \frac{c_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{c_{-1}}{(z-a)}$

3) a je bistvena, če ima glavni del neskončno členov

---

(14) Določi glavni del Laurentove vrste za
a) $f(z) = \frac{\cos z}{z - \sin z}$ okrog točke $a = 0$.
b) $f(z) = \frac{1}{(z+1)(e^{i\pi z}+1)}$ okrog točke $a = -1$.

(a):

$$f(z) = \frac{\cos z}{z - \sin z} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots)} = \frac{1 - \frac{z^2}{2} + \cdots}{\frac{z^3}{3!} - \frac{z^5}{5!} + \cdots} = \frac{1 - \frac{z^2}{2} + \cdots}{z^3(\frac{1}{3!} - \frac{z^2}{5!} + \cdots)}$$

$\neq 0$ za $z=0$

$\neq 0$ za $z=0$

pri $z=0$ imamo pol stopnje 3

$$f(z) = \underbrace{\frac{c_{-3}}{z^3} + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z}}_{\text{glavni del}} + \cdots \overset{\text{"odpravimo ulomke"}}{\Longrightarrow} 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots = \underbrace{\left(\frac{c_{-3}}{z^3} + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + \cdots\right) z^3}_{= c_{-3} + c_{-2}z + c_{-1}z^2 + \cdots}\left(\frac{1}{3!} - \frac{z^2}{5!} + \cdots\right)$$

pri z^0: $1 = \frac{c_{-3}}{3!} \to c_{-3} = 3! = 6$
pri z: $0 = \frac{c_{-2}}{3!} \to c_{-2} = 0^*$
pri z^2: $-\frac{1}{2!} = -\frac{c_{-3}}{5!} + \frac{c_{-1}}{3!} \to c_{-1} = -\frac{27}{10}$

* f je liha funkcija $(f(-z) = -f(z))$ $\Rightarrow$ vsi sodi koeficienti v Laurentovi vrsti so enaki 0.

$\to$ glavni del $= \frac{6}{z^3} - \frac{27}{10z}$

(b): Za razvoj okoli a želimo $f(z) = \sum_{-\infty}^{\infty} c_n (\overbrace{z-a}^{w})^n$.
Uvedemo $w = z-a$ in si pomagamo z razvojem okoli 0.

$$f(z) = \frac{1}{w(e^{i\pi(w-1)}+1)} = \sum_{-\infty}^{\infty} c_n w^n$$

$$e^{i\pi w} \underset{-1}{\underbrace{e^{-i\pi}}} + 1 = -\left(1 + i\pi w + \frac{(i\pi w)^2}{2!} + \cdots\right) + 1 = -i\pi w - \frac{(i\pi w)^2}{2!} - \cdots = -i\pi w + \frac{(\pi w)^2}{2} - \cdots$$

$$\Rightarrow f(z) = \frac{1}{w(-i\pi w + \frac{\pi^2 w^2}{2} - \cdots)} = \frac{1}{\underset{\uparrow \atop (z+1)^2}{w^2}(-i\pi + \frac{\pi^2}{2}w - \cdots)} = \underbrace{\frac{c_{-2}}{w^2} + \frac{c_{-1}}{w}}_{\text{glavni del}} + \cdots$$

$\Rightarrow$ v $a=-1$ ima f pol stopnje 2

$$\to 1 = \left(\frac{c_{-2}}{w^2} + \frac{c_{-1}}{w} + \cdots\right) w^2 \left(-i\pi + \frac{\pi^2}{2}w - \cdots\right) \Rightarrow \underset{w^0}{c_{-2}} = \frac{i}{\pi}\ ;\ \underset{w}{c_{-1}} = \frac{1}{2}$$

$\to$ glavni del $= \frac{i}{\pi w^2} + \frac{1}{2w}$

(22) Naj bo $-1 < \alpha < 1$ in $0 < \varepsilon < R$. S pomočjo kompleksne integracije S pomočjo kompleksne integracije po robu območja $D = \{z \mid |z| \in (\varepsilon, R)\,,\ \operatorname{Re} z > 0\}$izračunaj integral

$$I = \int_0^\infty \frac{x^\alpha dx}{1+x^2}$$

in nato za $0 < p < 1$ izpelji formulo

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}.$$

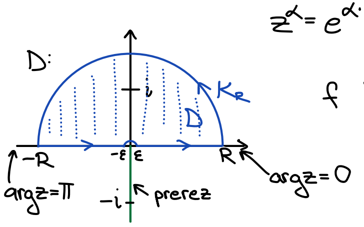
Nasvet: Za $0 < n+1 < m$ velja

$$\int_0^\infty \frac{x^n dx}{1+x^m} = \frac{1}{m}\Gamma(\frac{n+1}{m})\Gamma(1-\frac{n+1}{m}). = \frac{1}{m}B\left(\frac{n+1}{m}, 1-\frac{n+1}{m}\right)$$

$(\alpha = n,\ 2 = m)$

$$f(z) = \frac{z^\alpha}{1+z^2} = \frac{z^\alpha}{(z+i)(z-i)}$$

$\int_{\partial D} f(z)dz$ ← kaj D je ne vemo



$z^\alpha = e^{\alpha \cdot \log z}$ → izberemo vejo $\log z$: $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$

f holomorfna na $\overline{D} \setminus \{i\}$

$i^\alpha = e^{\alpha \log i} = e^{\alpha(\overset{0}{\log 1} + \frac{\pi}{2}i)} = e^{\alpha\frac{\pi}{2}i}$

$$\rightarrow \underline{\int_{\partial D} \frac{z^\alpha}{(z-i)(z+i)}dz} = \int_{\partial D} \frac{g(z)}{z-i}dz = 2\pi i\, g(i) = 2\pi i \frac{i^\alpha}{2i} = \pi i^\alpha = \underline{\pi e^{\alpha\frac{\pi}{2}i}}$$

$g(z) = \frac{z^\alpha}{z+i}$

$$\underline{\left|\int_{K_R} f(z)dz\right|} = \left|\int_0^\pi \frac{(Re^{i\varphi})^\alpha Re^{i\varphi} i}{1+(Re^{i\varphi})^2}d\varphi\right| \le \int_0^\pi |\ldots|\,d\varphi \le \int_0^\pi \frac{R^{\alpha+1}}{R^2-1}d\varphi = \pi\frac{R^{\alpha+1}}{R^2-1} \xrightarrow{R\to\infty} \underline{0}$$

$z = Re^{i\varphi}$
$dz = Rie^{i\varphi}d\varphi$

$(Re^{i\varphi})^\alpha = e^{\alpha\log(Re^{i\varphi})} = e^{\alpha(\log R + \varphi i)} = e^{\alpha \log R} \cdot e^{\alpha i\varphi} = R^\alpha e^{\alpha i \varphi}$ $|\ldots| = 1$

$|1+(Re^{i\varphi})^2| \ge |(Re^{i\varphi})^2| - 1 = R^2 - 1$

$$\underline{\int_{K_\varepsilon} f(z)dz} = \int_\pi^0 \underbrace{\frac{(\varepsilon e^{i\varphi})^\alpha \varepsilon e^{i\varphi} i}{1+(\varepsilon e^{i\varphi})^2}}_{h(\varepsilon,\varphi)\text{ je zvezna na }[0,\infty)\times[0,\pi]} d\varphi \xrightarrow{\varepsilon\to 0} \underline{0}$$

$\varepsilon^{\alpha+1} > 0$ ← zvezno za $\varepsilon \in [0,\infty)$

$$\underline{\int_\varepsilon^R f(z)dz} \xrightarrow[\varepsilon\to 0]{R\to\infty} \underline{I}$$

$(-t)^\alpha = e^{\alpha\log(-t)} = e^{\alpha(\log t + \overset{\arg(-t)}{i\pi})} = e^{\alpha\log t}e^{\alpha i\pi} = t^\alpha e^{\alpha i \pi}$

$$\underline{\int_{-R}^{-\varepsilon} f(z)dz} = \int_{-R}^{-\varepsilon}\frac{z^\alpha}{1+z^2}dz = \int_{-R}^{-\varepsilon}\frac{x^\alpha}{1+x^2}dx \xrightarrow[\varepsilon\to 0]{R\to\infty} \int_{-\infty}^0 \frac{x^\alpha}{1+x^2}dx = \int_0^\infty \frac{(-t)^\alpha}{1+t^2}dt = \int_0^\infty \frac{t^\alpha e^{\alpha i\pi}}{1+t^2}dt = e^{\alpha i \pi}\int_0^\infty \frac{t^\alpha}{1+t^2}dt =$$

$z = x$, $dz = dx$; $t = -x$

$$= \underline{e^{\alpha i\pi} \cdot I}$$

$$\Rightarrow \pi \cdot e^{\alpha\frac{\pi}{2}i} = \int_{\partial D} f(z)dz = \int_{K_R}\ldots + \int_{K_\varepsilon}\ldots + \int_\varepsilon^R\ldots + \int_{-R}^{-\varepsilon}\ldots \xrightarrow[\varepsilon\to 0]{R\to\infty} 0 + 0 + I + e^{\alpha i\pi}I$$

$$\Rightarrow \underline{I} = \pi\frac{e^{\alpha\frac{\pi}{2}i}}{1+e^{\alpha i\pi}} =$$

$$= \frac{\pi e^{\alpha\frac{\pi}{2}i}}{1+e^{\alpha i\pi}} \cdot \frac{e^{-i\alpha\frac{\pi}{2}}}{e^{-\alpha i\frac{\pi}{2}}} =$$

$$= \frac{\pi}{e^{-\alpha\frac{\pi}{2}i} + e^{\alpha i\frac{\pi}{2}}} = \underline{\frac{\pi}{2\cos(\alpha\frac{\pi}{2})}}$$

$e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi$
$e^{i\varphi} - e^{-i\varphi} = 2i\sin\varphi$

Po nasvetu je: $I = \frac{1}{2}\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{1-\alpha}{2})$ in $p = \frac{\alpha+1}{2}$ → $\alpha = 2p-1$

$$\Rightarrow \frac{\pi}{\cos((2p-1)\frac{\pi}{2})} = \Gamma(p)\Gamma(1-p)$$

$$\underline{\frac{\pi}{\sin p\pi} = \Gamma(p)\Gamma(1-p)}$$

(19) Prevedi integral

$$I = \int_0^{2\pi} h(\cos x, \sin x) dx$$

na integral kompleksne funkcije in nato za $a > 1$ izračunaj integral

$$\int_0^{2\pi} \frac{\cos x \, dx}{a - \cos x}.$$

$$\left.\begin{matrix} x \in [0, 2\pi] \\ \Updownarrow \\ z = e^{ix} \end{matrix}\right\} \to \text{integracija po enotski krožnici}$$

$dz = ie^{ix} dx \to dx = \frac{dz}{iz}$ "$\bar{z}$ prepovedana v holomorfnih funkcijah"

$$2i \sin x = z - \bar{z} \Rightarrow \sin x = \frac{z - \bar{z}}{2i} = \frac{z - z^{-1}}{2i}$$

$$2 \cos x = z + \bar{z} \Rightarrow \cos x = \frac{z + \bar{z}}{2} = \frac{z + z^{-1}}{2}$$

$$\Rightarrow I = \underbrace{\int_{|z|=1} \underbrace{h\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)} \frac{dz}{iz}}$$

Če je h "lepa" funkcija, je to holomorfna funkcija.

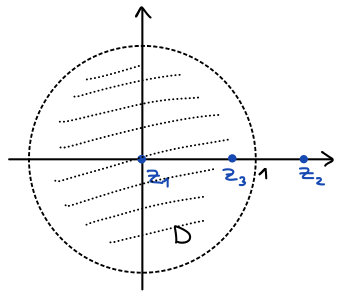
$$\int_0^{2\pi} \frac{\cos x \, dx}{a - \cos x} \underset{\uparrow}{=} \int_{|z|=1} \frac{z + z^{-1}}{2a - z - z^{-1}} \frac{dz}{iz} = \int_{|z|=1} \frac{z^2 + 1}{2az - z^2 - 1} \frac{dz}{iz} = I$$

$\cos x = \frac{z + z^{-1}}{2}$, $dx = \frac{dz}{iz}$

Razvoj: $f \in \mathcal{H}(\bar{D}^{\text{op}} \setminus A) \Rightarrow \int_{\partial D} f(z) dz = 2\pi i \sum_{a \in A} \text{Res}(f, a)$

residuum $= e_{-1}$
(koeficient pri $\frac{1}{z-a}$ v LV okoli a)

$D = \Delta(0, 1)$
$z_1 = 0$
$z_{2,3} = a \pm \sqrt{a^2 - 1}$



$$z^2 - 2az + 1 = (z - z_2)(z - z_3)$$

Viéte: $z_2 + z_3 = 2a$
$z_2 z_3 = 1 \longrightarrow |z_2| > 1 \Rightarrow |z_3| < 1$

$$I = 2\pi i \left[\text{Res}(f, z_1) + \text{Res}(f, z_3)\right]$$

Če je a pol stopnje n:

$$\text{Res}(f, a) = \frac{1}{(n-1)!} \lim_{z \to a} \left[f(z)(z - a)^n\right]^{(n-1)}$$

$$\underset{\substack{\uparrow \\ \text{pol stopnje 1}}}{\text{Res}(f, 0)} = 1 \cdot \lim_{z \to 0} (f(z) z) = \lim_{z \to 0} \frac{z^2 + 1}{i(z^2 + 2az - 1)} = -\frac{1}{i} = i$$

$$\underset{\substack{\uparrow \\ \text{pol stopnje 1}}}{\text{Res}(f, z_3)} = 1 \cdot \lim_{z \to z_3} \left[f(z)(z - z_3)\right] = \lim_{z \to z_3} \left[-\frac{z^2 + 1}{iz(z - z_2)}\right] = -\frac{z_3^2 + 1}{iz_3(z_3 - z_2)} = \frac{2az_3}{2\sqrt{a^2 - 1}(iz_3)} = \frac{a}{i\sqrt{a^2 - 1}}$$

$$\Rightarrow \underline{I} = 2\pi i \left(-\frac{1}{i} + \frac{a}{i\sqrt{a^2 - 1}}\right) = \underline{2\pi \left(\frac{a}{\sqrt{a^2 - 1}} - 1\right)}$$

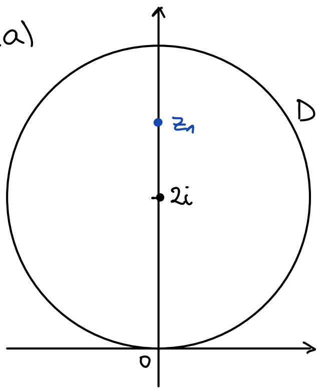
(17) Izračunaj kompleksne integrale

a) $\int_{|z-2i|=2} \frac{dz}{z^2+9}$.

b) $\int_{|z-i/2|=1} \frac{dz}{(z^2+1)^3}$.

c) $\int_{|z|=3} \sin \frac{z}{z+i} dz$.

(a)



$$\frac{1}{z^2+9} = \frac{1}{(z-3i)(z+3i)}$$

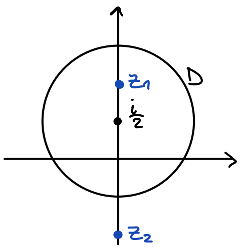
singularnosti: $z_1 = 3i$, $z_2 = -3i$

$\uparrow$ pol stopnje 1

$$\int_{|z-2i|=2} \frac{dz}{z^2+9} = 2\pi i \operatorname{Res}(f, 3i) = \frac{\pi}{3}$$

$$\operatorname{Res}(f, 3i) = 1 \cdot \lim_{z \to 3i} \left[\frac{1}{z^2+9}(z-3i)\right] = \lim_{z \to 3i} \frac{1}{z+3i} = \frac{1}{6i}$$

(b)



$$\frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$$

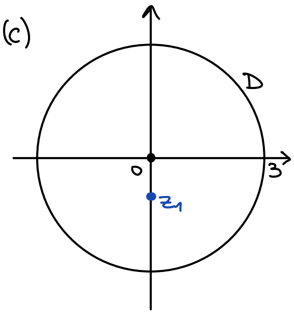
singularnosti: $z_1 = i$, $z_2 = -i$

pol stopnje 3

$$\int_{|z-\frac{i}{2}|=1} \frac{dz}{(z^2+1)^3} = 2\pi i \operatorname{Res}(f, i) = \frac{3\pi}{8}$$

$$\operatorname{Res}(f, i) = \frac{1}{2} \lim_{z \to i} \left[\frac{(z-1)^3}{(z^2+1)^3}\right]'' = \frac{1}{2} \lim_{z \to i} \left[\frac{1}{(z+i)^3}\right]'' = 6 \frac{1}{2^5 i^5} = \frac{3}{2^4 i}$$

(c)



$\frac{z}{z+i} \rightarrow$ singularnost: $z_1 = -i$

$$\int_{|z|=3} \sin \frac{z}{z+i} dz = 2\pi i \operatorname{Res}(f, -i) = 2\pi \cos 1$$

$$\operatorname{Res}(f, -i) = -i \cos 1$$

Residuum poiščemo z vrsto:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z+i)^k \leftarrow z+i = w \Rightarrow f(z) = \sin \frac{w+i}{w} = \sin\left(1 + \frac{i}{w}\right) = \sin 1 \cos \frac{1}{w} - \sin \frac{1}{w} \cos 1 =$$
$$= \sin 1 \cdot \left[1 - \frac{i^2}{2!\, w^2} + \frac{i^4}{4!\, w^4} - \cdots\right] - \cos 1 \left[\frac{1}{w} - \frac{i^3}{3!\, w^3} + \cdots\right]$$

Iščemo c_{-1}, to je koeficient pri w^{-1}: $c_{-1} = -i \cos 1 = \operatorname{Res}(f, -i)$

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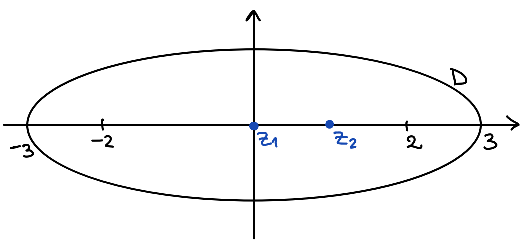
(18) Izračunaj kompleksni integral

$$\int_{|z-2|+|z+2|=6} \frac{1}{z} e^{\frac{1}{1-z}} dz.$$



singularnosti: $z_1 = 0 \leftarrow$ pol stopnje 1

$z_2 = 1$

$$\int_D \frac{1}{z} e^{\frac{1}{1-z}} dz = 2\pi i \left[\operatorname{Res}(f, 0) + \operatorname{Res}(f, 1)\right] = 2\pi i \left[e + \frac{1-e}{1-i}\right] = 2\pi i$$

$$\operatorname{Res}(f,0)=\lim_{z\to 0}\left(\frac{1}{z}e^{\frac{1}{1-z}}z\right)=e$$

$$\operatorname{Res}(f,1):\ w=z-1 \Rightarrow f(z)=\frac{1}{w+1}e^{-\frac{1}{w}}=\left(1-w+w^2-\dots\right)\left(1-\frac{1}{w}+\frac{1}{2!\,w^2}-\frac{1}{3!\,w^3}+\dots\right)=\dots+\frac{1}{w}\left(-1-\frac{1}{2!}-\frac{1}{3!}-\dots\right)+\dots=\dots+\frac{1-e}{w}+\dots$$

$$\frac{1}{1+w}=\frac{1}{1-(-w)}=1-w+w^2-\dots$$

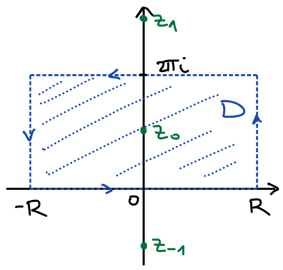
$$e^{-\frac{1}{w}}=1-\frac{1}{w}+\frac{1}{2!\,w^2}-\frac{1}{3!\,w^3}+\dots$$

Iščemo c_{-1}, to je koeficient pri w^{-1}: $c_{-1}=\frac{1-e}{1-i}=\operatorname{Res}(f,1)$

---

(26) Naj bosta $a\in(0,1)$ in $R>0$. S pomočjo kompleksne integracijje po robu območja $D=\{z \mid \operatorname{Re} z\in(-R,R),\ \operatorname{Im} z\in(0,2\pi)\}$ izračunaj integral

$$I=\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx.$$



$$I_{\mathbb{C}}=\int_{\partial D}f(z)dz\ ;\quad f(z)=\frac{e^{az}}{1+e^z}\ \to\ \text{singularnosti: } 1+e^z=0$$

$$z_k=i\pi+2k\pi i;\ k\in\mathbb{Z}$$

$$\to z_0=\pi i,\ z_1=3\pi i,\ z_{-1}=-\pi i$$

$$\Rightarrow I_{\mathbb{C}}=2\pi i\cdot\operatorname{Res}(f,i\pi)=-2\pi i\,e^{ai\pi}$$

z_0 stopnje 1; uporabimo L'Hôpitalovo pravilo

$$\operatorname{Res}(f,i\pi)\overset{z_0 \text{ stopnje } 1}{=}\lim_{z\to i\pi}\left[\frac{e^{az}}{1+e^z}(z-i\pi)\right]=e^{ai\pi}\lim_{z\to i\pi}\frac{z-i\pi}{1+e^z}=e^{ai\pi}\cdot\lim_{z\to i\pi}\frac{1}{e^z}=-e^{ai\pi}$$

1) $$\int_{-R}^{R}f(z)dz\overset{z=x}{=}\int_{-R}^{R}f(x)\,dx\xrightarrow{R\to\infty}I$$

2) $z=R+\pi i t;\ t\in(0,2);\ dz=\pi i\,dt$

$$\int_{R}^{R+2i\pi}f(z)dz=\pi i\int_0^2\frac{e^{a(R+i\pi t)}}{1+e^{R+\pi i t}}dt$$

$|e^{i\varphi}|=1$ za $\varphi\in\mathbb{R}$; $0\ (a<1)$

$$\to\ \left|\int_0^2\frac{e^{a(R+i\pi t)}}{1+e^{R+\pi i t}}dt\right|\le\int_0^2\left|\frac{e^{aR}e^{t\pi i}}{1+e^Re^{t\pi i}}\right|dt\le\int_0^2\frac{e^{aR}}{e^R-1}dt=\frac{2e^{aR}}{e^R-1}\cdot\frac{e^{-R}}{e^{-R}}=\frac{2e^{R(a-1)}}{1-e^{-R}}\xrightarrow{R\to\infty}0$$

$$|1+e^Re^{t\pi i}|\ge\underbrace{|e^Re^{t\pi i}|}_{e^R}-1$$

3) $z=x+2\pi i;\ x\in[-R,R]$; $dz=dx$

$$\int_{R+2\pi i}^{-R+2\pi i}f(z)dz=\int_R^{-R}\frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}dx=\int_R^{-R}\frac{e^{ax}e^{2\pi i a}}{1+e^x\underbrace{e^{2\pi i}}_{1}}dx=e^{2\pi i a}\int_R^{-R}\frac{e^{ax}}{1+e^x}dx\xrightarrow{R\to\infty}-e^{2\pi i a}I$$

4) $$\int_{-R+2\pi i}^{-R}f(z)dz\ \underset{(a>0)}{\xrightarrow{R\to\infty}}\ 0\qquad \text{(D.N)}$$

$$R\to\infty\ \Rightarrow\ -2\pi i e^{ai\pi}=I+0-e^{a2\pi i}I+0$$

$$\Rightarrow I=\frac{2\pi i\,e^{ai\pi}}{e^{a2\pi i}-1}\cdot\frac{e^{-a\pi i}}{e^{-a\pi i}}=\frac{2\pi i\cdot 1}{2i\sin(a\pi)}=\frac{\pi}{\sin(a\pi)}$$

$$\left(e^{i\varphi}+e^{-i\varphi}=2\cos\varphi\quad\text{in}\quad e^{i\varphi}-e^{-i\varphi}=2i\sin\varphi\right)$$

Kako bi določili stopnjo pola?

Z razvojem v vrsto: $1+e^z=1+e^{w+i\pi}=1+e^we^{i\pi}=1-e^w=1-\left(1+w+\frac{w^2}{2!}+\dots\right)=-w-\dots$

$e^{i\pi}=-1$

$$f(z)=\frac{e^{az}}{-w-\dots}=\frac{\dots}{w(-1-\dots)}$$

$e^{az}\neq 0$ za $z=i\pi$

pol stopnje 1

(28) Določi število ničel polinoma $p(z) = z^5 + 7z^4 - z^2 + 2$

a) na odprtem krogu $B(0,1)$

b) na zaprtem krogu $\overline{B}(0,1)$

c) na odprtem kolobarju $A(0;2,10)$

Rouchéjev izrek: $D^{odp} \subseteq \mathbb{C}$, omejeno območje s kosoma gladkim robom;
$f, F \in \mathcal{H}(\overline{D})$ in $\forall z \in \partial D.\ |f(z)| < |F(z)|$
$\Rightarrow$ število ničel za $F+f$ na D je enako številu ničel za F

(a) $D = \Delta(0,1)$

$z \in \partial D \Leftrightarrow |z| = 1$

Kaj je največji člen v $p(z)$ pri tem pogoju?

$F(z) = 7z^4 \to |F(z)| = 7$

$f(z) = z^5 - z^2 + 2 \to \underline{|f(z)|} \leq 1+1+2 = 4 < 7 = \underline{|F(z)|} \overset{R.I.}{\Rightarrow}$ $\underline{p \text{ ima } 4 \text{ ničle na } \Delta(0,1)}$

$0 \in \Delta(0,1)$ 4-kratna ničla

Dodatek k Rouchéjevem izreku: iz pogoja $z \in \partial D.\ |f(z)| < |F(z)|$ sledi
$|(F+f)(z)| = |F(z) + f(z)| \geq |F(z)| - |f(z)| > 0$
$\Rightarrow (F+f)(z) \neq 0 \Leftarrow F+f$ nima ničel na ∂D

(b) $D = \overline{\Delta(0,1)}$

$\Rightarrow$ $\underline{p \text{ ima } 4 \text{ ničle na } \overline{\Delta(0,1)}}$

(c) $D = A(0;2,10) = \Delta(0,10) \setminus \overline{\Delta(0,2)}$

Preštejemo ničle na $\Delta(0,10)$: $|z| = 10 \to F(z) = z^5 \to |F(z)| = 10^5 = 100.000$

$f(z) = 7z^4 - z^2 + 2 \to \underline{|f(z)|} \leq 70.102 < \underline{|F(z)|}$

$\Rightarrow$ p ima 5 ničel na $\Delta(0,10)$

Preštejmo ničle na $\Delta(0,2)$: $|z| = 2 \to F(z) = 7z^4 \to |F(z)| = 112$

$f(z) = z^5 - z^2 + 2 \to \underline{|f(z)|} \leq 2^5 + 2^2 + 2 = 38 < \underline{|F(z)|}$

$\Rightarrow$ p ima 4 ničle na $\Delta(0,2)$

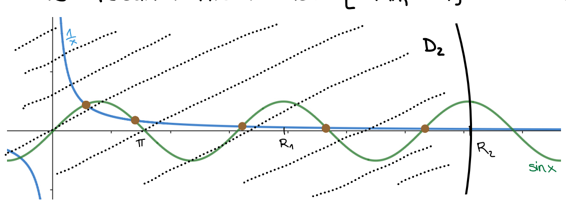
$\Rightarrow$ $\underline{\text{na } A(0;2,10) \text{ ima } p \text{ eno ničlo}}$

---

(29) Dokaži, da ima enačba $z \sin z = 1$ same realne rešitve.

Nasvet: Preštej ničle na krogu $|z| < 2n\pi + \frac{\pi}{2}$, $n \in \mathbb{N}$.

1) Število realnih ničel na $[-R_n, R_n]$: $x \sin x = 1 \to \sin x = \frac{1}{x}$



$\to 2(1+2n) = \#$ realnih ničel na D_n

simetrija preko izhodišča

2) Število kompleksnih ničel na D_n: $g(z) = z \sin z - 1$

Rouchéjev izrek: $F(z) = z \cdot \sin z$

$f(z) = -1$

$|F(z)| > |f(z)|$ za $|z| = R_n$

$\to |z \cdot \sin z| > 1$

$z = x + iy \to |(x+iy) \sin(x+iy)|^2 > 1$

$\sin(x+iy) = \sin x \cdot \cosh y + i \sinh y \cdot \cos x$

$|x+iy|^2 \cdot |\sin x \cdot \cosh y + i \sinh y \cdot \cos x|^2 = \underset{R_n^2 \text{ pri pogoju } |z| = R_n}{(x^2+y^2)}(\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x) > 1$

$\to \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x > \frac{1}{R_n^2}$

$\sin^2 x \cdot \cosh^2 y + \sinh^2 y (1 - \sin^2 x) > \frac{1}{R_n^2}$

$\sin^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y > \frac{1}{R_n^2}$

$\underbrace{\sin^2 x + \sinh^2 y}_{h(x,y)} > \frac{1}{R_n^2}$

Vezani ekstrem: $L(x,y) = h(x,y) + \lambda(x^2+y^2-R_n^2)$

$L_x(x,y) = 2\sin x \cdot \cos x + 2\lambda x = 0$

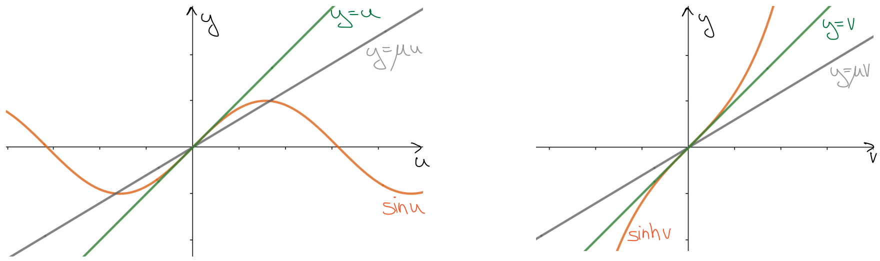
$L_y(x,y) = 2\sinh y \cdot \cosh y + 2\lambda y = 0$

$\to$ $\sin 2x = -\lambda \cdot 2x$ nove spremenljivke: $u = 2x$ in $\mu = -\lambda$

$\sinh 2y = -\lambda \cdot 2y$ $v = 2y$

$\to$ $\sin u = \mu u$

$\sinh v = \mu v$



$(\sinh x)' = \cosh x$

$(\sinh x)'(0) = 1 \to y = x$ (tangenta)

- Če je $\mu \leq 1$, ima enačba $\sinh v = \mu v$ samo eno rešitev: $v = 0$ (graf $\sinh x$ je konveksen na $(0,\infty)$).
- Če je $\mu > 1$, ima enačba $\sin u = \mu u$ samo eno rešitev: $u = 0$.

Kandidati za ekstreme: 1) $v = 0 \Rightarrow y = 0 \Rightarrow x = \pm R_n$

2) $u = 0 \Rightarrow x = 0 \Rightarrow y = \pm R_n$

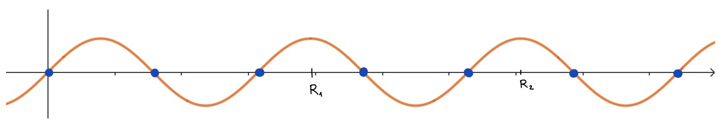
$h(0, \pm R_n) = \sinh^2(R_n) \underset{\substack{\uparrow \\ x \geq 0:\ \sinh x \geq x}}{\geq} R_n^2 > \frac{1}{R_n^2}$ $(R_n > 1,\ \forall n)$

$h(\pm R_n, 0) = \sin^2(R_n) = 1 > \frac{1}{R_n^2}$

$\overset{R.I.}{\Rightarrow}$ Število ničel $g(z) = z\cdot\sin z - 1$ na D_n je enako številu ničel $z\cdot\sin z$ na D_n.

Ničle $z\cdot\sin z$ na D_n: (DN: preveri, da ima $z\cdot\sin z = 0$ same realne ničle)

1) $\sin x = 0$:



$\# = 1 + 4n$ $(= 5 + 4(n-1))$

2) $x = 0$: pri $z = 0$ imamo dvojno rešitev

$\Rightarrow$ število vseh ničel $F(z)$ je $1 + 4n + 1 = 2(1+2n)$.

$\Rightarrow$ Število realnih ničel je enako številu vseh ničel.

$\Rightarrow$ vse rešitve enačbe $z\cdot\sin z = 1$ so realne

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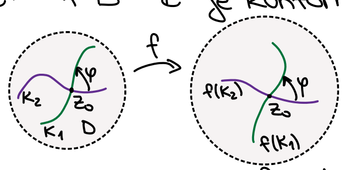
KONFORMNOST:

$D^{odp} \subseteq \mathbb{C}$: $f: D \to \mathbb{C}$ je konformna, če ohranja kote med krivuljami



konformnost $\Leftrightarrow$ holomorfnost $\wedge$ $f'(z) \neq 0$

BIHOLOMORFNA EKVIVALENCA:
$D_1^{odp}, D_2^{odp} \subseteq \mathbb{C}$ sta biholomorfno ekvivalentni, če obstaja holomorfna bijektivna preslikava $f: D_1 \to D_2$, ki ima holomorfen inverz.

RIEMANNOV UPODOBITVENI IZREK:
$D^{odp} \subseteq \mathbb{C}$ je biholomorfno ekvivalentna $\triangle$ $\Leftrightarrow$ D enostavno povezana in $D \neq \mathbb{C}$

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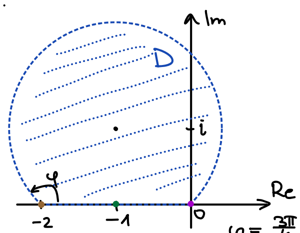
(30) Ugotovi ali je območje D možno z biholomorfno preslikavo $f : D \to \triangle$ preslikati na odprt enotski krog $\triangle$. V primerih, ko f obstaja, poišči kak primer take preslikave.

a) $D = \{z \mid |z - i + 1| < \sqrt{2},\ \operatorname{Im} z > 0\}$
b) $D = \{z \mid |z - i| > \frac{1}{2},\ |z - 2i| < 2\}$.
c) $D = \{z \mid |z - i| > 1,\ |z - 2i| < 2\}$.

(a):



$\varphi = \frac{3\pi}{4}$

D je po Riemannu biholomorfno ekvivalentna $\triangle$.

Namig: D najprej preslikaj z $f_1(z) = \frac{az+b}{cz+d}$ (lomljena linearna transformacija); $ad - bc \neq 0$

$f_1: \mathbb{CP}^1 \to \mathbb{CP}^1$
$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ ... Riemannova sfera

$f_1(-\frac{d}{c}) = \frac{\cdots}{0} := \infty$

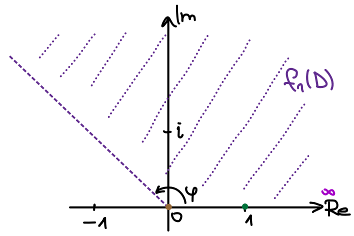
$f_1(\infty) := \frac{a}{c}$

f_1 slika premice in krožnice v premice in krožnice

$f_1(-2) = 0$
$f_1(0) = \infty$
$f_1(-1) = 1$ $\Big\} \to$ realna os $\mapsto$ realna os / $\mathbb{R} \mapsto \mathbb{R}$

$f_1(-2) = 0 \Rightarrow f_1(z) = \frac{a(z+2)}{\cdots}$ (ničla pri $z=-2$)
$f_1(0) = \infty \Rightarrow f_1(z) = \frac{a(z+2)}{cz}$ (pol pri $z=0$)
Lahko izberemo $c=1$: $f_1(z) = \frac{1}{z} a(z+2)$
$f_1(-1) = \frac{a \cdot 1}{-1} = -a = 1 \to a = -1$

$\Rightarrow$ $f_1(z) = \frac{-(z+2)}{z}$



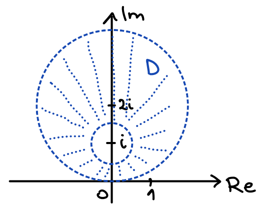
f_2 preslika D_1 v zgornjo polravnino
$f_2(z) = z^\alpha$, $\alpha > 0$
↑ kot se pomnoži s faktorjem α
želimo: $\alpha \cdot \frac{3\pi}{4} = \pi \to \alpha = \frac{4}{3}$
$\to f_2(z) = z^{\frac{4}{3}} = e^{\frac{4}{3}\log z}$
↑ velja: $\arg z \in (0, 2\pi)$

D_1 smo preslikali na $D_2 = \{z \mid \operatorname{Im} z > 0\}$

$f_0: D_2 \to \triangle$; $z \mapsto \frac{iz+1}{iz-1}$

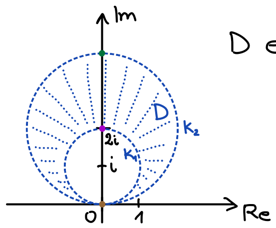
$\Rightarrow f = f_0 \circ f_2 \circ f_1$

(b):



D ni enostavno povezano
$\Rightarrow$ D ni biholomorfno ekvivalentno $\triangle$

(c):



D enostavno povezano $\wedge$ $D \neq \mathbb{C}$ $\Rightarrow$ D biholomorfno ekvivalentno $\triangle$

$f_1(z) = \frac{az+b}{cz+d}$: $f_1(0) = \infty \Rightarrow f_1(z) = \frac{az+b}{z} \to c=1, d=0$
$f_1(2i) = 0 \Rightarrow f_1(z) = \frac{a(z-2i)}{z} \to b=0$
$f_1(4i) = \pi i = \frac{a \cdot 2i}{4i} \to a = 2\pi i$
$\Rightarrow f_1(z) = \frac{2\pi i(z-2i)}{z}$

Kaj je slika Im?

Ker je $0 \in \mathrm{Im}$ in $f_1(0)=\infty$, je slika premica. $\Rightarrow f_1(\mathrm{Im})=\mathrm{Im}$

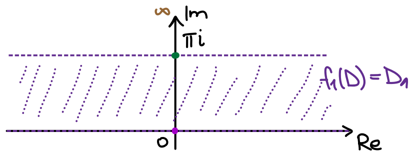
Kaj je slika K_1?

Ker je $0 \in K_1$ in $f_1(2i)=0$ ter $\varphi_{K_1,\mathrm{Im}}=\pi$, je slika premica. $\Rightarrow f_1(K_1)=\mathrm{Re}$ $\quad(\varphi_{f(K_1),\mathrm{Im}}=\pi)$

Kaj je slika K_2?

Ker je $0 \in K_2$ in $f_1(4i)=\pi i$ ter $\varphi_{K_1,\mathrm{Im}}=\pi$, je slika premica. $\Rightarrow f_1(K_2)=\mathrm{Re}+\pi i$

$\Rightarrow f_1(D)=\{z \mid 0<\mathrm{Im}\, z<4i\}$



$f_2: z \mapsto e^z = e^x e^{iy}$

$z \in D_1 \Leftrightarrow x \in \mathbb{R},\ y \in (0,\pi) \Leftrightarrow e^x \in (0,\infty) \wedge e^{iy} \in \{w \mid |w|=1 \wedge \mathrm{Im}\, w>0\}$

$\rightarrow f_2(D_1)=H$ (zgornja polravnina)

$f_3: H \rightarrow \triangle\ ;\ z \mapsto \frac{iz+1}{iz-1}$

$\Rightarrow f = f_3 \circ f_2 \circ f_1$

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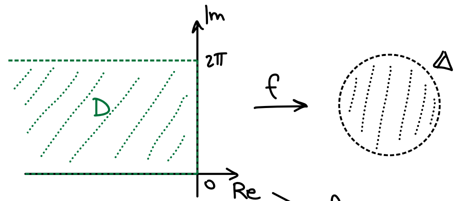
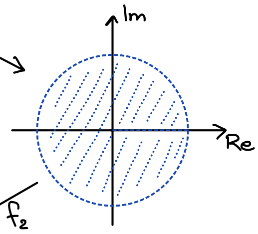
(31) Poišči kako biholomorfno preslikavo $f : D \to \triangle$, če je
a) $D = \{z \mid \mathrm{Re}\, z < 0\,,\ \mathrm{Im}\, z \in (0, 2\pi)\}$.
b) $D = \{z \mid |z-1| > 1\,,\ |z-2| < 2\,,\ \mathrm{Im}\, z \in (0, 2\pi)\}$.

(a):



f_1



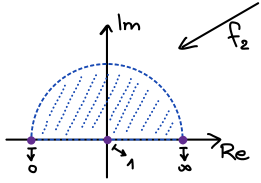
$f_1(z)=e^z$

$f_1(D)$: $\quad e^{x+iy} = e^{x}e^{iy}$, $\quad x<0$, $\quad e^{x} \in (0,1)$, $\quad e^{iy} \in (0,2\pi)$

f_2

$f_2(z)=\sqrt{z}$

$f_2(f_1(D))$:



f_3

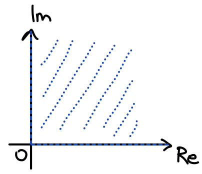
$f_3(z) = \frac{az+b}{cz+d}$

$-1 \mapsto 0$

$0 \mapsto 1$

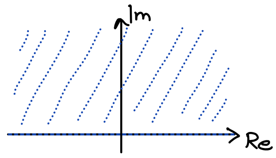
$1 \mapsto \infty \quad \rightarrow \quad f_3(z)=\frac{a(z+1)}{z-1}$

$f(0)=1=-a \rightarrow a=-1 \quad \Rightarrow f_3(z)=\frac{z+1}{1-z}$



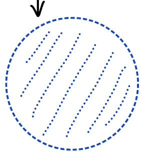
f_4

$f_4(z)=z^2$



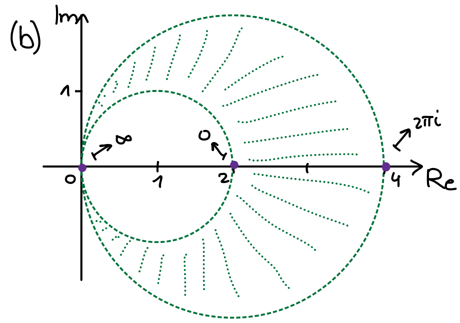
f_0

$f_0(z)=\frac{iz+1}{iz-1}$

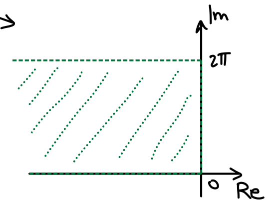


$f = f_0 \circ f_4 \circ f_3 \circ f_2 \circ f_1$

(b)



f_5



$f_5(z)=\frac{az+b}{cz+d}$

$0\mapsto\infty \quad \rightarrow \frac{a(z-2)}{z}$

$2\mapsto 0 \quad \Rightarrow f_5(4)=2\pi i=\frac{a}{2}\rightarrow a=4\pi i$

$4\mapsto 2\pi i \quad \Rightarrow \underline{f_5(z)=4\pi i\,\frac{z-2}{z}}$

Nadaljujemo po a): $\underline{F=f\circ f_5}$

X: Poišči vse pare celih funkcij $f,g\in\mathcal{H}(\mathbb{C})$, ki zadoščajo: $f(z)^2+g(z)^2=1$ za $\forall z\in\mathbb{C}$.

Opazimo možnost: $\left.\begin{array}{l} f(z)=\cos z\\ g(z)=\sin z\end{array}\right\}\Rightarrow \begin{array}{l}\cos(h(z))\\ \sin(h(z))\end{array}$; h cela, $h\in\mathcal{H}(\mathbb{C})$.

$\underbrace{(f(z)+ig(z))}_{k(z)\,;\,k\in\mathcal{H}(\mathbb{C})}(f(z)-ig(z))=1$

nima ničel

IZREK: Če je $k\in\mathcal{H}(U)$, kjer je U enostavno povezana odprta množica in k nima ničel, potem obstaja $h\in\mathcal{H}(U)$, da je $k(z)=e^{h(z)}$.
(Obstaja $h(z)=\log(k(z))$.)

Obstaja $h(z)$, $h\in\mathcal{H}(\mathbb{C})$, da je $k(z)=e^{ih(z)}\overset{(1)}{=}f(z)+ig(z)$.

$\Rightarrow f(z)-ig(z)\overset{(2)}{=}e^{-ih(z)}$

$e^{ih(z)}=\cos(h(z))+i\sin(h(z))$
$e^{-ih(z)}=\cos(h(z))-i\sin(h(z))$

$(1)+(2)\rightarrow\ e^{ih(z)}+e^{-ih(z)}=2f(z)=2\cos(h(z))\ \Rightarrow f(z)=\cos(h(z))$
$(1)-(2)\rightarrow\ e^{ih(z)}-e^{-ih(z)}=2ig(z)=2i\sin(h(z))\ \Rightarrow \underline{g(z)=\sin(h(z))}$

Y: Naj bo f holomorfna na povezani odprti okolici točke $\infty\in\mathbb{CP}^1$, za katero je $f(n)\overset{(*)}{=}\frac{n^2\sqrt[n]{e}}{1+n^2}$ pri dovolj velikih številih n. Določi f.

Ponuja se: $f(z)=\frac{z^2\sqrt[z]{e}}{1+z^2}$

f holomorfna v $\infty \overset{def}{\Leftrightarrow} g(z)=f(\frac{1}{z})$ holomorfna v 0

$(*)\Rightarrow f(n)=g(\frac{1}{n})=\frac{n^2\sqrt[n]{e}}{1+n^2}=\frac{n^2e^{\frac{1}{n}}}{1+n^2}=\frac{(\frac{1}{n})^{-2}e^{\frac{1}{n}}}{1+(\frac{1}{n})^{-2}}=h(\frac{1}{n})$ $\qquad \frac{1}{n}\rightsquigarrow z:\ h(z)=\frac{R^{-2}e^{z}}{1+R^{-2}}=\frac{e^{z}}{R^{2}+1}$

h holomorfna v okolici 0:
$g=h$ na množici $\{\frac{1}{n}\,|\,n\geq N\}$, ki ima stekališče 0, ki je v območju holomorfnosti

$\Rightarrow g\equiv h:\qquad \underline{f(z)=g(\frac{1}{z})=\frac{e^{\frac{1}{z}}}{\frac{1}{z^2}+1}}=\underline{\frac{z^2e^{\frac{1}{z}}}{1+z^2}}$