Analiza 2b (PREDAVANJA)

FOURIEROVE VRSTE

HILBERTOV PROSTOR:

X... vektorski prostor nad $\mathbb{R}$ $(\mathbb{C})$

SKALARNI PRODUKT: $\langle\cdot,\cdot\rangle : X\times X \to \mathbb{R}$

Za $x,y,z\in X$ in $\lambda,\mu\in\mathbb{R}$ velja:
(1): $\langle x,x\rangle \geq 0$
(2): $\langle x,x\rangle = 0 \Leftrightarrow x=0$
} POZITIVNA DEFINITNOST
(3): $\langle x,y\rangle = \langle y,x\rangle$ ← SIMETRIČNOST (nad $\mathbb{C}$: $\langle x,y\rangle = \overline{\langle y,x\rangle}$)
(4): $\langle \lambda x+\mu y, z\rangle = \lambda\langle x,z\rangle + \mu\langle y,z\rangle$ ← LINEARNOST V 1. FAKTORJU

CAUCHY-SCHWARZOVA NEENAKOST: $|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\cdot\sqrt{\langle y,y\rangle}$

Nad $\mathbb{R}$: $\|x\| = \sqrt{\langle x,x\rangle}$; $|\langle x,y\rangle| = \|x\|\cdot\|y\|$

kvadratna enačba

$t \mapsto \langle x+ty, x+ty\rangle = \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle \geq 0$

$\hookrightarrow D = 4\langle x,y\rangle^2 - 4\langle x,x\rangle\langle y,y\rangle \leq 0 \Leftrightarrow |\langle x,y\rangle| \leq \|x\|\cdot\|y\|$

Nad $\mathbb{C}$: $\langle x,y\rangle = \alpha|\langle x,y\rangle|$; $\alpha\in\mathbb{C} \wedge |\alpha|=1$

$\bar{\alpha}\langle x,y\rangle = |\langle x,y\rangle|$ in $\overline{\langle \alpha y, x\rangle} = \langle x,\alpha y\rangle = |\langle x,y\rangle|$

$t \mapsto \langle x+t\alpha y, x+t\alpha y\rangle = \langle x,x\rangle + t\langle \alpha y,x\rangle + t\langle x,\alpha y\rangle + t^2\langle y,y\rangle =$
$= \|x\|^2 + 2t\cdot|\langle x,y\rangle| + t^2\|y\|^2$

NORMA na X: $\|x\| = \sqrt{\langle x,x\rangle}$

Za vsak $x,y\in X$ in $\lambda\in\mathbb{R}$ velja:
(1): $\|x\|\geq 0$
(2): $\|x\| = 0 \Leftrightarrow x=0$
(3): $\|\lambda x\| = |\lambda|\cdot\|x\|$
(4): $\|x+y\| \leq \|x\| + \|y\|$

Dokaz (4): $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2$ $\square$

DEFINICIJA: $(X,\langle\cdot,\cdot\rangle)$ je HILBERTOV PROSTOR, če je poln metrični prostor v metriki porojeni iz $\langle\cdot,\cdot\rangle$.

PRIMERI:
(1): $\mathbb{R}^n$: $x\cdot y = x_1y_1 + \dots + x_ny_n$; $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$

$\|x\| = \sqrt{x_1^2+\dots+x_n^2} \to d_2(x,y) = \sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2}$

$\Rightarrow (\mathbb{R}^n,\cdot)$ Hilbertov prostor

(2): $\mathbb{C}^n$: $z\cdot w = z_1\overline{w_1} + \dots + z_n\overline{w_n}$; $z=(z_1,\dots,z_n)$, $w=(w_1,\dots,w_n)$

$\|z\|_2 = \sqrt{|z_1|^2+\dots+|z_n|^2} \to d(z,w) = \sqrt{|z_1-w_1|^2+\dots+|z_n-w_n|^2}$

(3): $X = C[a,b]$ ... realne funkcije

$f,g \in X: \quad \langle f,g\rangle = \int_a^b f(x)g(x)\,dx$

(Če f,g kompleksni funkciji $/f = u+iv/$: $\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$.)

$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$ je skalarni produkt na $C[a,b]$

(1): $\langle f,f\rangle \geq 0$
(3): $\langle f,g\rangle = \langle g,f\rangle$
(4): $\langle \lambda f + \mu g, h\rangle = \lambda\langle f,h\rangle + \mu\langle g,h\rangle$

} očitno iz lastnosti integrala

Denimo: $0 = \int_a^b f^2(x)\,dx$

Če $f \neq 0 \Rightarrow \langle f,f\rangle = \int_a^b f^2(x)\,dx > 0$

$\Downarrow\ \exists x_0 \in [a,b].$
$f(x_0) \neq 0 \Rightarrow f^2(x_0) > 0$

f je zvezna $\Rightarrow \exists \delta > 0.\ \forall x \in [a,b] \cap [x_0-\delta, x_0+\delta] = I \ni: f^2(x) \geq \frac{f(x_0)}{2}$.

Torej: $\int_a^b f^2(x)\,dx \geq \int_I \frac{f^2(x_0)}{2} = \frac{f^2(x_0)}{2}\,\ell(I) > 0$

$\rightarrow \int_a^b f^2(x)\,dx = 0 \Leftrightarrow f^2 = 0$ s.p
$\Leftrightarrow f = 0$ s.p. $\Rightarrow f = 0$

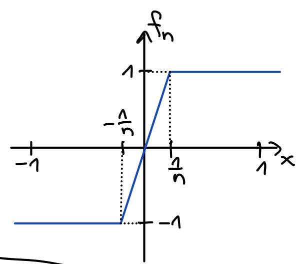
(množice z mero 0 imajo prazno notranjost)

$\Rightarrow$ $(C[a,b], \langle\cdot,\cdot\rangle)$ ni Hilbertov prostor

$\rightarrow d_2(f,g) = \sqrt{\int_a^b [f(x)-g(x)]^2\,dx}$

<u>$(C[-1,1], \langle\cdot,\cdot\rangle)$ ni Hilbertov</u>

$$f_n(x) = \begin{cases} 1; & \frac{1}{n} < x \leq 1 \\ nx; & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ -1; & -1 \leq x < -\frac{1}{n} \end{cases} \quad ; \quad f_n \in C[-1,1]$$



$n \leq m$: $d(f_n,f_m) = \sqrt{\int_{-1}^{1}(f_n(x)-f_m(x))^2\,dx} = \sqrt{\int_{-\frac{1}{n}}^{\frac{1}{n}}(f_n(x)-f_m(x))^2\,dx} \leq 2\sqrt{\frac{2}{n}}$

Cauchyjevo zaporedje

Če ima to zaporedje limito f v $C[-1,1]$, potem $f(x) = 1;\ 0 < x \leq 1$
$f(x) = -1;\ -1 \leq x < 0$

taka funkcija ni zvezna v 0

Denimo, da za $x_0 \in (0,1)$ velja $f(x_0) \neq 1$, potem $|f(x_0)-1| > 0$.

Ker je f zvezna: $\exists \delta > 0.\ |f(x)-1| \geq \left|\frac{f(x_0)-1}{2}\right|$

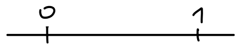
Za $x \in [x_0-\delta, x_0+\delta] \subseteq [0,1]$, kjer $0 < x_0 < 1$ in za dovolj velike n: $f_n(x) = 1$ na $[x_0-\delta, x_0+\delta]$.

$\frac{1}{n} < x_0-\delta$: $\rightarrow d(f,f_n) = \sqrt{\int_{-1}^{1}(f(x)-f_n(x))^2\,dx} \geq \sqrt{\int_{x_0-\delta}^{x_0+\delta}(f(x)-1)^2\,dx} \geq \sqrt{2\delta}\,\frac{f(x_0)-1}{2} \leq 2\sqrt{\frac{2}{n}}$?

$\square$?

$(M,d) \subseteq (\overline{M}, \overline{d})$ ... obstaja napolnitev metričnega prostora

$(\overline{M}, \overline{d})$ je poln metrični prostor; M je gosta v $\overline{M}$

Primer: $M=(0,1)$ z d_2 ; $\overline{M}=[0,1]$ z d_2 

DO TUKAJ PREPISANO

S kvadratom integrabilne funkcije:

$f:[a,b] \longrightarrow \mathbb{R}$:

$$\int_a^b |f(x)|^2 dx < \infty$$

$$L^2([a,b]) = \{ f:[a,b] \to \mathbb{R} ; \int_a^b |f(x)|^2 dx < \infty \}$$

$\Rightarrow$ $C[a,b] \subseteq L^2[a,b]$
odsekoma zvezne

$L^2([a,b])$ je vektorski prostor

$$\int_a^b (\lambda f)^2 dx = \lambda^2 \int_a^b f^2 dx < \infty \ ; \ \lambda \in \mathbb{R}$$

$$|f \cdot g| = \frac{1}{2}(|f|^2 + |g|^2) \Rightarrow \int_a^b |f(x)\, g(x)|\, dx \leq \frac{1}{2}\left(\int_a^b |f(x)|^2 dx + \int_a^b |g(x)|^2 dx \right)$$

$f,g \in L^2([a,b]) \Rightarrow |f \cdot g| \in L^2([a,b])$

$$\int_a^b (f+g)^2 dx = \int_a^b (f^2 + 2fg + g^2)\, dx < \infty$$

$f,g \in L^2([a,b]) \Rightarrow$ integral končen

$f,g \in L^2([a,b])$:

$$\langle f,g \rangle = \int_a^b f(x) g(x)\, dx \quad \leftarrow \text{obstaja}$$

$$\langle f,f\rangle = 0 = \int_a^b f(x)^2\,dx \iff f^2=0 \text{ s.p.} \iff f=0 \text{ s.p.}$$

$f=g$ v $L^2([a,b])$ oz. $f=g$ s.p.

DEJSTVO: $(L^2([a,b]), \langle\cdot,\cdot\rangle)$ je napolnitev $(C([a,b]), \langle\cdot,\cdot\rangle)$

$f\in L^2([a,b])$: obstaja zaporedje $f_n \in C[a,b]$; da je $\lim_{n\to\infty} f_n = f$ v $L^2([a,b])$

$\forall \varepsilon>0.\ \exists g\in C[a,b]$:
$$d_2(f,g) = \sqrt{\int_a^b (f(x)-g(x))^2\,dx} < \varepsilon.$$

$(X, \langle\cdot,\cdot\rangle)$ ... splošen vektorski prostor s skalarnim produktom

pravokotnost: $x,y\in X$: $x\perp y \iff \langle x,y\rangle = 0$

$\uparrow$

x je pravokoten na y

$x\perp y \iff y\perp x$

TRDITEV: (PITAGOROV IZREK)

Naj bodo $x_1,\dots,x_n\in X$ parowa pravokotni vektorji: $x_i\perp x_j$; $i\neq j$.

Tedaj:
$$\|x_1+\dots+x_n\|^2 = \|x_1\|^2+\dots+\|x_n\|^2$$

DOKAZ:
$$\|x_1+\dots+x_n\|^2 = \langle x_1+\dots+x_n, x_1+\dots+x_n\rangle = \sum_i\sum_j \langle x_i, x_j\rangle = \|x_1\|^2+\dots+\|x_n\|^2 \quad \square$$

Naj bo $Y\subseteq X$ podprostor; $x\in X$

Pravokotna projekcija vektorja x na podprostor Y, če obstaja, je tak vektor $P_Y(x)\in Y$, da je $x-P_Y(x)$ pravokoten na vsak vektor iz Y.

Če je $A\subseteq X$ neka podmnožica, potem definiramo ORTOGONALNI KOMPLEMENT:
$$A^\perp = \{x\in X,\ x\perp a \text{ za } \forall a\in A\}$$

TRDITEV: $A^\perp$ je vektorski podprostor X.

Dokaz:

$x\in A^\perp$, $\lambda\in\mathbb{R}$, $a\in A$
$$\langle \lambda x, a\rangle = \lambda\langle x,a\rangle = 0 \Rightarrow \lambda x\in A^\perp$$

$x,y\in A^\perp$, $a\in A$:
$$\langle x+y, a\rangle = \langle x,a\rangle + \langle y,a\rangle = 0+0 = 0 \quad \square$$

Opomba: 1) $(A^\perp)^\perp \supseteq A$

2) A podprostor; v splošnem velja $A \subseteq (A^\perp)^\perp$

$X = L^2([a,b])$; $A = C([a,b])$:

$A^\perp = \{0\}$; $(A^\perp)^\perp = X$

3) $A \subseteq (X, \langle\cdot,\cdot\rangle)$. A zaprt podprostor

Naj bo $x_0 = \lim_{n\to\infty} x_n$; $x_n \in A^\perp$.

$$a \in A \Rightarrow \langle x_n, a\rangle = 0 \quad \forall n$$

$$|\langle x_n, a\rangle - \langle x_0, a\rangle| = |\langle x_n - x_0, a\rangle| \leq \|x_n - x_0\| \cdot \|a\|$$

Velja: $0 = \lim_{n\to\infty} \langle x_n, a\rangle = \langle \lim_{n\to\infty} x_n, a\rangle = \langle x_0, a\rangle = 0 \Rightarrow x_0 \in A^\perp$

Opomba: $(X, \langle\cdot,\cdot\rangle)$ Hilbertov: V zaprt podprostor $(V^\perp)^\perp = V$

PRAVOKOTNA PROJEKCIJA:

$(X, \langle\cdot,\cdot\rangle)$, $Y \subseteq X$ podprostor ; $x \in X$

Pravokotna projekcija X na Y (če obstaja) je tak vektor $P_Y(x) \in Y$, da velja: $x - P_Y(x) \in Y^\perp$

TRDITEV: Naj bo $(X, \langle\cdot,\cdot\rangle)$ vektorski prostor s skalarnim produktom. Naj bo $Y \subseteq X$ podprostor in $x \in X$. Če obstaja pravokotna projekcija X na Y, potem je enolično določena in je najboljša aproksimacija vektorja x z vektorji v Y:

$$\|x - P_Y(x)\| = \min_{y \in Y} \|x - y\|$$

Velja tudi: $\|P_Y(x)\| \leq \|x\|$ in enakost velja $\Leftrightarrow$ $X = Y$

ZGLED: $Y = C([a,b]) \subseteq L^2([a,b]) = X$

Vektorji iz $L^2([a,b]) \setminus Y$ nimajo pravokotne projekcije na Y

Dokaz:

$y_1, y_2 \in Y$ pravokotni projekciji x na Y.

$$x - y_1, x - y_2 \in Y^\perp \Rightarrow (x - y_1) - (x - y_2) = y_2 - y_1 \in Y^\perp$$

$$y_2 - y_1 \in Y \quad (Y \text{ podprostor})$$

$$\Rightarrow y_2 - y_1 \perp y_2 - y_1 \Leftrightarrow y_1 - y_2 = 0$$
$$\Leftrightarrow y_1 = y_2$$

$$w \in Y: \quad x - w = \underbrace{(x - P_Y(x))}_{\in Y^\perp} + \underbrace{(P_Y(x) - w)}_{\in Y}$$

Pitagorov izrek:

$$\|x-w\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x) - w\|^2 \geq \|x - P_Y(x)\|^2$$

$$\Rightarrow \min_{y \in Y} \|x-w\| = \|x - P_Y(x)\|$$

$$w=0: \quad \|x\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x)\|^2 \geq \|P_Y(x)\|^2$$

$\square$

Opomba: Če je P_Y definirana na X, je linearen operator:

$$\sup_{\|x\|=1} \|P_Y(x)\| = \|P_Y\| = 1 \quad \text{za} \quad Y \neq \{0\}$$

(Predpostavka je izpolnjena, če je X Hilbertov, Y zaprt podprostor.)

Dokaz:

$x \in X, \lambda \in \mathbb{R}: \quad P_Y(\lambda x) = \lambda P_Y(x)$

$$\lambda x - \lambda P_Y(x) = \lambda \underbrace{(x - P_Y(x))}_{\in Y^\perp} \in Y^\perp \quad \Rightarrow \quad \lambda P_Y(x) = P_Y(\lambda x)$$

$$x_1, x_2 \in X: \quad (x_1 + x_2) - (P_Y(x_1) + P_Y(x_2)) = \underbrace{(x_1 - P_Y(x_1))}_{\in Y^\perp} + \underbrace{(x_2 - P_Y(x_2))}_{\in Y^\perp} \in Y^\perp$$

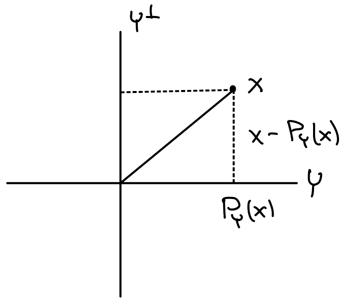
$$\Rightarrow \quad P_Y(x_1 + x_2) = P_Y(x_1) + P_Y(x_2)$$

Opomba: $P_Y^2 = P_Y$

Opomba: Če ima x pravokotno projekcijo na Y, potem ima pravokotno projekcijo tudi na $Y^\perp$.

Dokaz:

$P_Y(x) \in Y$



Projekcija x na $Y^\perp$ je vektor $x - P_Y(x) \in Y^\perp$

Če $x - (x - P_Y(x)) = P_Y(x) \in (Y^\perp)^\perp \quad \Rightarrow \quad P_Y(x) \in Y \subseteq (Y^\perp)^\perp$

$\square$

TRDITEV: Naj bo $(X, \langle\cdot,\cdot\rangle)$ vektorski prostor s skalarnim produktom.
Naj bo $Y \subseteq X$ končno dimenzionalni vektorski podprostor z ortonormirano bazo $\{e_1, \dots, e_n\}$.
Za $x \in X$ je pravokotna projekcija x na Y podana kot:
$$P_Y(x) = \sum_{i=1}^{n} \langle x, e_i\rangle e_i.$$

Dokaz:
Očitno: $\sum_{i=1}^{n} \langle x, e_i\rangle e_i \in Y$

Poglejmo $x - P_Y(x) = x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i$

$j_0 \in \{1, \dots, n\}$:
$$\langle x - P_Y(x), e_{j_0}\rangle = \langle x, e_{j_0}\rangle - \langle x, e_{j_0}\rangle\langle e_{j_0}, e_{j_0}\rangle = 0$$
$$\Rightarrow x - P_Y(x) \in Y^{\perp} \quad \square$$

POSLEDICA: Za vse podprostore $Y \subseteq X$, ki so končne kodimenzije, tudi obstaja pravokotna projekcija za $\forall x \in X$.

$(X, \langle\cdot,\cdot\rangle)$ vektorski prostor s skalarnim produktom:

ORTOGONALEN SISTEM (OS) v $(X, \langle\cdot,\cdot\rangle)$: $\{e_j\}_{j=1}^{\infty} \subseteq X \quad e_j \perp e_i \quad i \neq j$

ORTONORMIRAN SISTEM (ONS) v $(X, \langle\cdot,\cdot\rangle)$: $\{e_j\}_1^{\infty} \quad \langle e_i, e_j\rangle = \delta_{ij} = \begin{cases} 1 ; & i = j \\ 0 ; & i \neq j \end{cases}$

$x \in X$: glede na ONS:
$\{\langle x, e_j\rangle\}$ FOURIEROVI KOEFICIENTI x glede na $\{e_j\}_1^{\infty}$

TRDITEV (Bessellova neenakost)
Naj bo $(X, \langle\cdot,\cdot\rangle)$ vektorski prostor s skalarnim produktom.
Naj bo $\{e_j\}_1^{\infty}$ ortonormiran sistem in $x \in X$.
Tedaj velja:
$$\sum_1^{\infty} |\langle x, e_j\rangle|^2 \leq \|x\|^2$$

Dokaz:
$Y_n = \mathcal{L}(\{e_1, \dots, e_n\})$; $\{e_1, \dots, e_n\}$ je ortonormirana baza Y

$x \in X$:
$$P_Y(x) = \sum_1^{n} \langle x, e_j\rangle e_j$$
$$\|P_{Y_n}(x)\|^2 = \sum_1^{n} |\langle x, e_j\rangle|^2 \leq \|x\|^2$$
$$\Rightarrow \sum_1^{\infty} |\langle x, e_j\rangle|^2 \leq \|x\|^2 \quad \square$$

POSLEDICA: $\lim_{j\to\infty} \langle x, e_j\rangle = 0$

Naj bo $(X, \langle\cdot,\cdot\rangle)$ Hilbertov in $\{c_j\}_1^\infty$ taka števila, da velja $\sum_1^\infty |c_j|^2 < \infty$.

Naj bo $\{e_n\}_1^\infty$ ortonormiran sistem v $(X, \langle\cdot,\cdot\rangle)$.

TRDITEV: Obstaja vektor $x \in X$, za katerega velja $\langle x, e_n\rangle = c_n \ \forall n$.

Dokaz:

$$S_n = \sum_1^n c_j e_j \in X$$

$$s_n = \sum_1^n |c_j|^2$$

$n \in \mathbb{N}$, $p \in \mathbb{N}$:

$$\| S_{n+p} - S_n \|^2 = \| \sum_{n+1}^{n+p} c_j e_j \|^2 \overset{\text{PITAGORA}}{=} \sum_{n+1}^{n+p} |c_j|^2 = s_{n+p} - s_n$$

To zaporedje delnih vsot je konvergentno $\Rightarrow$ je Cauchyjevo.

$\Rightarrow \{S_n\}$ je Cauchyjevo
(ima limito)

$$x = \sum_1^\infty c_j e_j = \lim_{n\to\infty} \sum_1^n c_j e_j$$

$$\langle x, e_m\rangle = \langle \sum_1^\infty c_j e_j , e_m\rangle = \langle \lim_{n\to\infty} \sum_1^n c_j e_j , e_m \rangle = \lim_{n\to\infty} \langle \sum_1^n c_j e_j , e_m\rangle = c_m \qquad \square$$

$(X, \langle\cdot,\cdot\rangle)$ Hilbertov prostor, $(e_n)_n$ ortonormiran sistem

$x \in X \rightsquigarrow (\langle x, e_n\rangle)_n \qquad \sum_1^\infty |\langle x, e_n\rangle|^2 \le \|x\|^2 < \infty$

Lahko tvorimo $\tilde{x} = \sum_1^\infty \langle x, e_n\rangle e_n$

Opomba: $\{e_n\}_1^\infty$ ortonormiran sistem
$\forall x$, je $x = \tilde{x} = \sum_1^\infty \langle x, e_n\rangle e_n$

DEFINICIJA: Ortonormiran sistem $(e_j)_{j=1}^\infty$ je KOMPLETEN OZ. POLN SISTEM (KONS), če za vsak $x \in X$ velja:

$$x = \sum_1^\infty \langle x, e_j\rangle e_j$$

Primer: $\{1, \cos(nx), \sin(nx)\}$, $n \in \mathbb{N}$ ortogonalen sistem $L^2([-\pi,\pi])$ / $L^2([0,2\pi])$

$\{ \frac{1}{2\sqrt{\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}} \sin(xn)\}$; $n \in \mathbb{N}$ ortonormiran sistem $L^2([-\pi,\pi])$

Zapis Fourierove vrste nad $\mathbb{R}$:

$$f(x) = a_0 + \sum_1^\infty (a_n \cos(nx) + a_n \sin(nx)) \qquad \text{(KONS)}$$

nad $\mathbb{C}$: $\{ \frac{1}{\sqrt{2\pi}} e^{inx} \}$; $n \in \mathbb{Z}$

(KONS):

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = \int_{-\pi}^{\pi} e^{i(n-m)x}\, dx = \begin{cases} \xrightarrow{n=m} 2\pi = \|e^{inx}\|^2 \\ \xrightarrow{n\neq m} \frac{1}{i(n-m)} e^{i(n-m)x} \Big|_{-\pi}^{\pi} = 0 \end{cases}$$

IZREK: Naj bo $(X, \langle \cdot,\cdot \rangle)$ Hilbertov prostor in $(e_n)_1^\infty$ ortonormiran sistem. Naslednje izjave so ekvivalentne:

1) $(e_n)_{n=1}^\infty$ je kompleten sistem: $\forall x \in X: \ x = \sum_1^\infty \langle x, e_j \rangle e_j$

2) $\forall x,y \in X: \ \langle x,y \rangle = \sum_{j=1}^\infty \langle x, e_j \rangle \langle y, e_j \rangle$ (nad $\mathbb{C}$: $\sum_1^\infty \langle x, e_j \rangle \overline{\langle y, e_j \rangle}$)

3) $\forall x \in X: \ \|x\|^2 = \sum_1^\infty |\langle x, e_j \rangle|^2$ (PARSEVALOVA ENAKOST)

4) $(e_n)_{n=1}^\infty$ ni vsebovan v nobenem strogo večjem ortonormiranem sistemu

5) Edini vektor, ki je pravokoten na vse $(e_n)_{n=1}^\infty$ je vektor 0.

6) Končne linearne kombinacije $(e_n)_{n=1}^\infty$ so goste v X.
($x \in X$, $\varepsilon > 0$: $\|x - \sum_1^N \lambda_j e_j\| < \varepsilon$)

Dokaz:

$(1) \Rightarrow (2)$: $x, y \in X$:

$$x = \sum_1^\infty \langle x, e_j \rangle e_j$$

$$\langle x,y \rangle = \left\langle \sum_1^\infty \langle x, e_j \rangle e_j, y \right\rangle = \left\langle \lim_{N\to\infty} \sum_1^N \langle x, e_j \rangle e_j, y \right\rangle =$$
$$= \lim_{N\to\infty} \left\langle \sum_1^N \langle x, e_j \rangle e_j, y \right\rangle = \lim_{N\to\infty} \sum_1^N \langle x, e_j \rangle \langle e_j, y \rangle =$$
$$= \sum_1^\infty \langle x, e_j \rangle \langle e_j, y \rangle$$

$(2) \Rightarrow (3)$: $x = y$: $\|x\|^2 = \sum_1^\infty |\langle x, e_j \rangle|^2$

$(3) \Rightarrow (4)$: Denimo, da obstaja strogo večji ortonormiran sistem:

$\exists e_0 \perp e_j \ \forall j$: $\|e_0\| = 1$ $\quad \|e_0\| = 1 = \sum_1^\infty |\langle e_0, e_j \rangle|^2 = 0$ $\times$

$\{e_n\}_{n=1}^\infty \cup \{e_0\}$ (takega e_0 ni)

$(4) \Rightarrow (5)$: Naj bo $x \perp e_j$ za vsak j.
- $\cdot$) $x = 0$ ✓
- $\cdot\cdot$) $x \neq 0$: $e_0 = \frac{x}{\|x\|} \Rightarrow \{e_j\}_1^\infty \cup \{e_0\}$ je strogo večji ortonormiran sistem $\times$

$\Rightarrow x = 0$

$(5) \Rightarrow (1)$: $x \in X \rightsquigarrow (\langle x, e_n \rangle)_{n=1}^\infty$

$$\tilde{x} = \sum_1^\infty \langle x, e_n \rangle e_n$$

$$v = x - \tilde{x} = x - \sum_1^\infty \langle x, e_j \rangle e_j \overset{(5)}{\Rightarrow} v = 0 \Leftrightarrow x = \tilde{x} = \sum_1^\infty \langle x, e_j \rangle e_j$$

$\Rightarrow \underline{v \perp e_j \ \forall j}$

$$\langle v, e_n \rangle = \left\langle x - \sum_1^\infty \langle x, e_j \rangle e_j, e_n \right\rangle = \langle x, e_n \rangle - \left\langle \sum_1^\infty \langle x, e_j \rangle e_j, e_n \right\rangle =$$
$$= \langle x, e_n \rangle - \sum_1^\infty \langle x, e_j \rangle \langle e_j, e_n \rangle = \langle x, e_n \rangle - \langle x, e_n \rangle = 0 \Rightarrow v = 0 \Leftrightarrow x = \tilde{x}$$

$(1) \Rightarrow (6)$: $\quad \forall x = \sum_1^\infty \langle x, e_j\rangle e_j = \lim_{N\to\infty} \sum_1^N \langle x, e_j\rangle e_j$

$\forall \varepsilon > 0 .\ \exists N_0 .\ \forall N \geq N_0 .\ \| x - \sum_1^N \langle x, e_j\rangle e_j \| < \varepsilon .$

$(6) \Rightarrow (5)$: $\quad x \perp e_j \ \forall j \Rightarrow x = 0$

Naj bo $\varepsilon > 0 . \Rightarrow N \in \mathbb{N} \quad \lambda_1, \dots, \lambda_N$

$\| x - \sum_1^N \lambda_j e_j \| < \varepsilon$

$\|x\|^2 = \langle x, x\rangle = \langle x - \sum_1^N \lambda_j e_j, x\rangle \Leftarrow \|x\| \leq \|x - \sum_1^\infty \lambda_j e_j\|$

$\leq \| x - \sum_1^N \lambda_j e_j \| \cdot \|x\|$

$\Rightarrow \|x\| < \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow \|x\| = 0 \Rightarrow x = 0$

$\square$

KLASIČNE FOURIEROVE VRSTE

$f, g \in L^2(-\pi, \pi) = \{ f : [-\pi, \pi] \to \mathbb{R} ; \int_{-\pi}^{\pi} |f|^2 \, dx < \infty \}$

$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx$

ORTONORMIRAN SISTEM: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) ; n \in \mathbb{N} \right\}$

$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos(nx) + b_n \sin(nx)$

$\forall \varepsilon > 0 .\ \exists N_0 .\ \forall N \geq N_0 .\ \| f - (\frac{a_0}{2} + \sum_1^N a_n \cos(nx) + b_n \sin(nx)) \|_2 < \varepsilon$

$\Leftrightarrow \sqrt{\int_{-\pi}^{\pi} | f(x) - (\frac{a_0}{2} + \sum_1^\infty a_n \cos(nx) + b_n \sin(nx))|^2 \, dx} < \varepsilon$

V tem kontekstu $\forall f \in L^2(-\pi, \pi)$ gledamo kot periodično funkcijo s periodo 2π.

$\frac{a_0}{2} \int_{-\pi}^{\pi} 1 \, dx = \int_{-\pi}^{\pi} f(x) \, dx$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$

$\int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = a_n \int_{-\pi}^{\pi} \cos^2(nx) \, dx = \pi a_n$

$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx ; \ n \geq 0 \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx ; \ n \geq 0 \end{aligned}}$$

KLASIČNI FOURIEROVI KOEFICIENTI

$n>0$: $\quad a_0 = \sqrt{\frac{2}{\pi}} \langle f, \frac{1}{\sqrt{2\pi}} \rangle$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{\sqrt{\pi}}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\cos(nx)}{\sqrt{\pi}}\,dx = \frac{1}{\sqrt{\pi}}\langle f, \frac{\cos(nx)}{\sqrt{\pi}}\rangle$$

$$b_n = \frac{1}{\sqrt{\pi}}\langle f, \frac{\sin(nx)}{\sqrt{\pi}}\rangle$$

$$a_0 = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\,dx = \frac{\sqrt{2\pi}}{\pi}\int_{-\pi}^{\pi} f(x)\frac{1}{\sqrt{2\pi}}\,dx$$

POSLEDICA (RIEMANN-LEBESGUOVA LEMA): $\lim_{n\to\infty} a_n = 0 \wedge \lim_{n\to\infty} b_n = 0$

POSLEDICA: $\left(\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\right)$; $n\in\mathbb{N}$ je kompleten sistem

Velja Parsevalova enakost:

$$\|f\|^2 = |\langle f, \frac{1}{\sqrt{2\pi}}\rangle|^2 + \sum_1^\infty |\langle f, \frac{\cos(nx)}{\sqrt{\pi}}\rangle|^2 + |\langle f, \frac{\sin(nx)}{\sqrt{\pi}}|^2 =$$

$$\boxed{= \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_1^\infty \left(|a_n|^2 + |b_n|^2\right)}$$

$n>0$: $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\cos(nx)\,dx = \frac{1}{\pi n}\sin(nx)\Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin(nx)\,dx = \frac{(-1)}{\pi n}\cos(nx)\Big|_0^{\pi} = \frac{-1}{\pi n}\left((-1)^n - 1\right) = \begin{cases} 0 & ; n \text{ sodo} \\ \frac{2}{4\pi} & ; n \text{ liho}\end{cases}$$

Zgled:

$$f(x) = \begin{cases} 1 & ; 0 \le x < \pi \\ 0 & -\pi \le x < 0 \end{cases}$$

$$b_n = \frac{1}{\pi n}\left(1 - (-1)^n\right) = \begin{cases} 0 & ; n \text{ sodo} \\ \frac{2}{\pi n} & ; n \text{ liho}\end{cases} \quad ; \quad a_0 = 1$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin(nx) =$$

$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)}\sin((2\pi+1)x)$$

$\uparrow$

L^2 KONV $\quad f(0) = 1 \quad \frac{1}{2} \leftarrow x=0$

$\qquad\qquad$ LS $\qquad$ DS

Parsevalova enakost: $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = 1$

$$1 = \frac{1}{2} + \sum_0^\infty \frac{4}{\pi^2(2n+1)^2}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} + \dots =$$

$$= \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \cdot \left(\frac{1}{2^2} + \frac{1}{4^2} + \dots\right) =$$

+?

$$= \frac{\pi^2}{8} + \frac{1}{4} S$$

$$\Rightarrow \boxed{S = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}$$

$f: [-\pi, \pi]$ + 2π periodična na $\mathbb{R}$

<u>ODSEKOMA ZVEZNA:</u> Na vsakem končnem intervalu ima največ končno mnogo točk nezveznosti.
V vsaki točki ima levo in desno limito.

<u>ODSEKOMA ODVEDLJIVA FUNKCIJA:</u> Na vsakem končnem intervalu je funkcija odvedljiva povsod, razen morda v končno mnogo točkah.
V vsaki točki ima levi in desni odvod.

IZREK: Naj bo $f: \mathbb{R} \to \mathbb{R}$ 2π periodična funkcija, ki je odsekoma zvezna in odsekoma odvedljiva funkcija.
Tedaj za vsak $x_0 \in \mathbb{R}$ velja:

leva limita

desna limita

$$\frac{f(x_0+) + f(x_0-)}{2} = \frac{a_0}{2} + \sum_{n}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Opomba: V dokazu tega se ne potrebuje, da je $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx)\right\}$ kompleten sistem.

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)} \sin((2n+1)x):$$

$x = 0$: $f(0+) = 1$, $f(0-) = 0$

$$\frac{f(0+) + f(0-)}{2} = \frac{1}{2}$$

$x = \frac{\pi}{2}$: $$1 = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin\left((2k+1)\frac{\pi}{2}\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

POMOŽNE UGOTOVITVE: (1) f periodična s periodo p; f integrabilna na vsakem končnem intervalu:

$$\forall a \in \mathbb{R}: \int_a^{a+p} f(t)\,dt = \int_0^p f(t)\,dt$$

Dokaz: $$\int_a^{a+p} f(t)\,dt = \int_a^p f(t)\,dt + \int_p^{a+p} f(t)\,dt \underset{t = x+p}{=} \int_a^p f(t)\,dt + \int_0^a f(x)\,dx = \int_0^p f(t)\,dt.$$ $\square$

(2) $$\underline{\frac{1}{2} + \sum_{1}^{n} \cos(kx)} = \frac{1}{2} + \sum_{1}^{n} \frac{e^{ikx} + e^{-ikx}}{2} = \frac{1}{2}\left(e^{-inx} + e^{-i(n-1)x} + \dots + e^{-ix} + 1 + e^{ix} + \dots + e^{inx}\right) =$$

$$= \frac{1}{2} e^{-inx}\left(1 + e^{ix} + e^{2ix} + \dots + e^{2inx}\right) = \frac{1}{2} e^{-inx} \frac{1 - e^{(2n+1)ix}}{1 - e^{ix}} = \frac{1}{2} \frac{e^{(n+1)ix} - e^{-inx}}{e^{ix} - 1} \cdot \frac{e^{-\frac{ix}{2}}}{e^{-\frac{ix}{2}}} =$$

$$= \frac{1}{2} \frac{e^{(n+\frac{1}{2})ix} - e^{-i(n+\frac{1}{2})x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}} = \underline{\frac{1}{2} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})} = D_n(x)} \leftarrow \text{DIRICHLETOVO JEDRO}$$

i) $D_n(x)$ je soda funkcija (kvocient lihih funkcij)

ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = 1 = \frac{2}{\pi}\int_{0}^{\pi} D_n(x)\,dx$

iii) $D_n(x) = \frac{1}{2}\frac{\sin(nx)}{\sin\frac{x}{2}}\cdot\cos\frac{x}{2} + \frac{\cos(nx)}{2}$

Dokaz (izrek): $f:\mathbb{R}\to\mathbb{R}$ 2π-periodična, odsekoma zvezna, odsekoma odvedljiva

$x_0 \in \mathbb{R}$:

$$S_n(x_0) = \frac{a_0}{2} + \sum_{1}^{n} a_k\cos(kx_0) + b_k\sin(kx_0) =$$

$$= \frac{a_0}{2} + \sum_{1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt\cdot\cos(kx_0) + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt\cdot\sin(kx_0)$$

$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$

$$\left.\begin{aligned}
&\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\big(\cos(kt)\cos(kx_0) + \sin(kt)\sin(kx_0)\big)\,dt = \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(k(t-x_0))\,dt = \\
&= \frac{1}{\pi}\int_{-\pi-x_0}^{\pi-x_0} f(y+x_0)\cos(ky)\,dy = \quad \leftarrow \text{periodičnost } 2\pi \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y+x_0)\cos(ky)\,dy
\end{aligned}\right\} \text{k-ti člen}$$

$$S_n(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y+x_0)\Big(\frac{1}{2} + \sum_{1}^{n}\cos(ky)\Big)\,dy = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y+x_0)D_n(y)\,dy \underset{(*)}{\xrightarrow[n\to\infty]{?}} \frac{f(x_0-)+f(x_0-)}{2}$$

$$S_n(x_0) = \frac{1}{\pi}\Big(\int_{0}^{\pi} f(y+x_0)D_n(y)\,dy + \int_{-\pi}^{0} f(y+x_0)D_n(y)\,dy\Big) =$$

$$= \frac{1}{\pi}\Big(\int_{0}^{\pi} f(y+x_0)D_n(y)\,dy + \int_{0}^{\pi} f(-y+x_0)D_n(y)\,dy\Big)$$

$$S_n(x_0) - \frac{f(x_0-)+f(x_0+)}{2} =$$

$$= \frac{1}{\pi}\Big(\int_{0}^{\pi} f(x_0+y)D_n(y)\,dy + \int_{0}^{\pi} f(x_0-y)D_n(y)\,dy\Big) - \frac{1}{\pi}\Big(\int_{0}^{\pi} f(x_0+)D_n(y)\,dy + \int_{0}^{\pi} f(x_0-)D_n(y)\,dy\Big) =$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\big(f(x_0+y)-f(x_0+)\big)D_n(y)\,dy + \frac{1}{\pi}\int_{0}^{\pi}\big(f(x_0-y)-f(x_0-)\big)D_n(y)\,dy$$

I. ČLEN:

$$\frac{1}{\pi}\int_{0}^{\pi}\big(f(x_0+y)-f(x_0+)\big)D_n(y)\,dy =$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\frac{f(x_0+y)-f(x_0+)}{y}\Big(\frac{y}{\sin\frac{y}{2}}\cdot\cos\frac{y}{2}\cdot\sin(ny) + y\cos(ny)\Big)\,dy$$

$$\int_{0}^{\pi}\underbrace{\frac{f(x_0+y)-f(x_0+)}{y}\,\frac{y}{\sin\frac{y}{2}}\cos\frac{y}{2}}_{F(y)}\sin(ny)\,dy = \int_{0}^{\pi}\underset{\in L^2(-\pi,\pi)}{F(y)}\sin(ny)\,dy \xrightarrow[n\to\infty]{} 0$$

$$F(y) = \begin{cases} 0 & ; -\pi \le y \le 0 \\ \frac{f(x_0+y)-f(x_0+)}{y}\cdot\frac{y}{\sin\frac{y}{2}}\cos\frac{y}{2} & ; 0 < y \le \pi \end{cases}$$

$\Rightarrow F(y) \in L^2(-\pi,\pi)$

$\Rightarrow \lim_{n\to\infty}\int_{0}^{\pi} F(y)\sin(ny)\,dy = 0$

(enako za ostale člene)

$\square$

Opombe:

(1) f soda $\Rightarrow b_n = 0$
f liha $\Rightarrow a_n = 0$

(2) $f: [0,\pi] \to \mathbb{R}$ lahko na $[-\pi,\pi]$ sodo razširimo: $f(x) = f(-x)$ ali
liho razširimo: $f(x) = -f(-x)$

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos(nx)$$

$$f(x) = \sum_{1}^{\infty} b_n \sin(nx)$$

Opomba: (*)

$$S_n(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(t)\, D_n(x-t)\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, D_n(x-t)\, dt$$

KONVOLUCIJA: f, g:

$$(f * g)(x) = \int_{-\pi}^{\pi} f(t) g(x-t)\, dt = \int_{-\pi}^{\pi} f(x-s) g(s)\, ds$$

$x - t = s \Rightarrow t = x - s$

Opomba: $[-\ell, \ell]$:
$$\left\{ \frac{1}{\sqrt{2\ell}}, \frac{1}{\sqrt{\ell}} \cos\left(n\pi \frac{x}{\ell}\right), \frac{1}{\sqrt{\ell}} \sin\left(n\pi \frac{x}{\ell}\right) \right\}; \; n \in \mathbb{N}$$

$$\int_{-\ell}^{\ell} \underbrace{\cos^2\left(n\pi \frac{x}{\ell}\right)}_{= \frac{1 + \cos(2n\pi \frac{x}{\ell})}{2}} = \ell \qquad \int_{-\ell}^{\ell} 1\, dx = 2\ell$$

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos\left(n\pi \frac{x}{\ell}\right) + b_n \sin\left(n\pi \frac{x}{\ell}\right)$$

Opomba: f periodična s periodo 2π, k-krat odvedljiva na $\mathbb{R}$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\, dt \underset{\text{PER PARTES}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\, \frac{1}{n} \sin(nt)\, dt$$

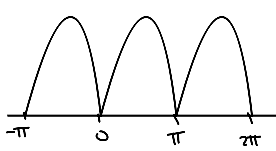
$du = f(t)dt$
$v = \frac{1}{n}\sin(nt)$

Če je $f \in C^k(\mathbb{R})$ in periodična s periodo 2π, potem velja:
$$a_n = O\left(\frac{1}{n^k}\right) = b_n$$

$k = 2$: $\underbrace{\frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos(nx) + b_n \sin(nx)}_{= f(x)}$, konvergira absolutno za $\forall x \in \mathbb{R}$ in enakomerno na $\mathbb{R}$ proti f.

Zgled:

$f(x) = \pi^2 - x^2$ $\leftarrow$ soda funkcija $\Rightarrow b_n = 0 \;\; \forall n$



zvezna ✓
odsekoma odvedljiva ✓

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\, dx = \frac{1}{\pi}\left(2\pi^3 - \frac{2}{3}\pi^3\right) = \frac{4}{3}\pi^2 \Rightarrow a_0^2 = \frac{16\pi^4}{9}$$

$$n \in \mathbb{N}: \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\cos(nx)\, dx = \frac{2}{\pi}\int_{0}^{\pi} (\pi^2 - x^2)\cos(nx)\, dx = \frac{2}{\pi}\cdot 2\int_{0}^{\pi} \frac{x}{n}\sin(nx)\, dx =$$

$u = x$
$v = -\frac{1}{n}\cos(nx)$

$$= \frac{4}{\pi n}\left[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\underbrace{\int_0^{\pi}\cos(nx)\,dx}_{=0}\right] = \frac{4\pi}{n^2\pi}(-1)(-1)^n = \frac{4}{n^2}(-1)^{n+1}$$

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_1^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos(nx)$$

$x=\pi$: $\quad 0 = \frac{2\pi^2}{3} + \sum_1^{\infty}\frac{4}{n^2}(-1) \quad\rightarrow\quad \boxed{\sum_1^{\infty}\frac{1}{n} = \frac{\pi^2}{6}}$

$x=0$: $\quad \frac{\pi^2}{3} = 4\sum_1^{\infty}\frac{(-1)^{n+1}}{n^2} \quad\rightarrow\quad \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

Parsevalova enakost: $\quad \frac{8\pi^4}{9} + 16\sum_1^{\infty}\frac{1}{n^4} = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)^2\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^4 - 2\pi^2x^2 + x^4)\,dx =$

$$= \frac{2}{\pi}\left(\pi^5 - \frac{2}{3}\pi^5 + \frac{1}{5}\pi^5\right) = \frac{16}{15}\pi^4$$

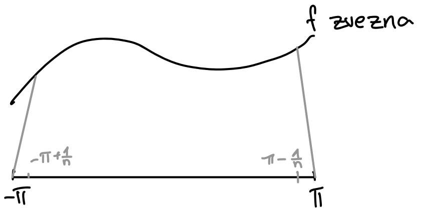
$$\rightarrow \boxed{\sum_1^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}}$$

CESÁROVE VSOTE:

Naj bo f zvezna periodična funkcija s periodo 2π. $f \in C[-\pi,\pi]$, $f(-\pi) = f(\pi)$.

$$\boxed{\sigma_N(x) = \frac{1}{N}\left(S_0(x) + \dots + S_{N-1}(x)\right)}$$

IZREK: (▲) Naj bo f zvezna periodična funkcija s periodo 2π ($f \in C[-\pi,\pi]$ in $f(-\pi) = f(\pi)$ + nadaljevanje s periodo 2π).
Tedaj Cesárove vsote konvergirajo k f enakomerno na $\mathbb{R}$ (oz. $[-\pi,\pi]$)



$f \in C[-\pi,\pi]$

f_n zvezne, periodične s periodo 2π
$f_n(\pi) = f_n(-\pi)$

$f_n \xrightarrow{L^2} f$

$$f_n(x) = \begin{cases} f(x) & ; \ -\pi + \frac{1}{n} \leq x \leq \pi - \frac{1}{n} \\ -nf(\pi - \frac{1}{n})(x - \pi) & ; \ \pi - \frac{1}{n} \leq x \leq \pi \\ nf(-\pi + \frac{1}{n})(x + \pi) & ; \ -\pi \leq x \leq -\pi + \frac{1}{n} \end{cases}$$

$$d(f_n, f)^2 = \|f_n - f\|^2 = \int_{-\pi}^{\pi}|f_n(x) - f(x)|^2\,dx = \int_{-\pi}^{-\pi+\frac{1}{n}}|f_n(x) - f(x)|^2 dx + \int_{\pi-\frac{1}{n}}^{\pi}|f_n(x) - f(x)|^2 dx =$$

$$\leq \frac{2}{n}4\cdot\|f\|_{\infty}^2$$

$f \in L^2[-\pi,\pi]$: $\exists$ zaporedje zveznih funkcij $g_n \xrightarrow{L^2} f$:
$\forall \varepsilon > 0.\ \exists g$ zvezna na $[-\pi,\pi]$. $d_2(f,g) < \varepsilon$

za g zvezna in $\varepsilon > 0$ obstaja $\tilde{g}$ zvezna periodična (2π), da $d_2(\tilde{g}, g) < \varepsilon$.

za $f \in L^2(-\pi,\pi)$ in $\varepsilon > 0$ obstaja zvezna periodična $\tilde{g}$ funkcija, da je $d_2(f,\tilde{g}) < 2\varepsilon$

Po izreku za funkcijo $\tilde{g}$ in $\varepsilon>0$ obstaja TRIGONOMETRIČNI POLINOM $T(x)$ (končna linearna kombinacija $\sin(nx), \cos(nx)$), da $\|\tilde{g}-T\|_\infty < \varepsilon$

$$d_2(\tilde{g},T) = \sqrt{\int_{-\pi}^{\pi} |\tilde{g}-T|^2 dx} < \varepsilon\sqrt{2\pi} \quad \Rightarrow \quad d_2(f,T) < \varepsilon(2+\sqrt{2\pi})$$

Trigonometrični polinomi so gosti v $L^2(-\pi,\pi)$.

$$\Rightarrow \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\right\},\ n\in\mathbb{N} \text{ je POLN SISTEM}$$

$f\in L^2(-\pi,\pi)$, $\varepsilon>0$: $\quad \exists\, g$ zvezna: $d_2(g,f)<\varepsilon$

$\exists\, \tilde{g}$ zvezna periodična: $d_2(\tilde{g},g)<\varepsilon$

izrek: $\exists T.\ d_\infty(T,\tilde{g})<\varepsilon \Rightarrow d_2(T,\tilde{g})<\varepsilon\sqrt{2\pi}$

Zgled:

$1-1+1-1+1-1+\cdots$

$S_1=1$
$S_2=0$
$S_3=1$
$\vdots$

$$\frac{1}{N}(S_1+\cdots+S_N) \overset{N=2k}{=} \frac{\frac{N}{2}}{N} = \frac{1}{2} \xrightarrow{N\to\infty} \frac{1}{2}$$

$$\overset{N=2k+1}{=} \frac{\frac{N-1}{2}+1}{N}$$

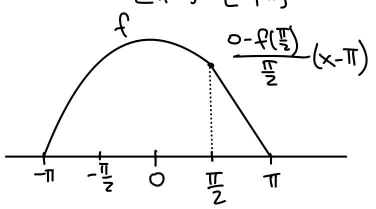
WEIERSTRASSOV IZREK: $[a,b]\in\mathbb{R}$, $f\in C[a,b]$, $\varepsilon>0$

$\forall f\in C[a,b].\ \exists$ polinom p: $\|f-p\|_{max} = \max_{[a,b]}|f(x)-p(x)|<\varepsilon$

$\mathcal{P}$ ... vsi polinomi $\subseteq C[a,b]$; $\overline{\mathcal{P}} = C[a,b]$

Dokaz:

BŠS: $[a,b]=[-\pi,\pi]$



$f\in C[-\frac{\pi}{2},\frac{\pi}{2}]$

Razširimo do zvezne, 2π periodične funkcije na $[-\pi,\pi]$

$\varepsilon>0$:

Po izreku obstaja T trigonometrični polinom: $\|f-T\|_{max\,[-\pi,\pi]}<\varepsilon$

$$T(x) = \alpha_0 + \sum_1^\infty \alpha_n\cos(nx) + \beta_n\sin(nx)$$

Vse $\cos(nx), \sin(nx)$, $n=1,\dots,N$, lahko na $[-\pi,\pi]$ poljubno dobro enakomerno aproksimiramo s trigonometričnimi polinomi: $\|T-p\|_\infty<\frac{\varepsilon}{2}$

POMOŽNA TRDITEV: $F_N(x) = \frac{1}{N}\sum_0^{N-1} D_i(x)$ ← FEJÉRJEVO JEDRO

i) $F_N(x) = \frac{1}{2N}\left(\frac{\sin(\frac{Nx}{2})}{\sin\frac{x}{2}}\right)^2$

ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1$

iii) $F_N(x)\geq 0 \quad \forall x$

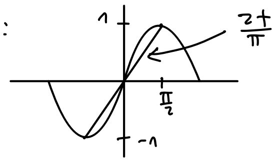
iv) $\forall a\in(0,\pi).\ \lim_{N\to\infty} F_N(x)=0$ enakomerno na $a\leq|x|\leq\pi$

Dokaz:

(ii) sledi iz $\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1$ + definicija F_N

(iii) sledi iz (i)

(i) $\Rightarrow$ (iv):



$0 \le |t| \le \frac{\pi}{2}$

$\frac{2|t|}{\pi} \le |\sin t| \to \frac{1}{|\sin t|} \le \frac{\pi}{2|t|}$

$t = \frac{x}{2}$, $0 \le |x| \le \pi \to \frac{1}{|\sin \frac{x}{2}|} \le \frac{\pi}{|x|}$

$0 \le a < |x| \le \pi$

$\frac{1}{|\sin \frac{x}{2}|} \le \frac{\pi}{|x|} \le \frac{\pi}{a} \Rightarrow |F_N(x)| \le \frac{1}{2N}\frac{\pi^2}{a^2} \longrightarrow 0$

enakomerno na $0 < a \le |x| \le \pi$

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y)\,D_N(y)\,dy = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,D_N(x-t)\,dt$$

$D_N(x) = \frac{1}{2}\,\frac{\sin((N+\frac{1}{2})x)}{\sin\frac{x}{2}}$ $\qquad$ $D_N(x)$ soda ; $\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1$

(i)
$$F_N(x) = \frac{1}{2N}\sum_{k=0}^{N-1}\frac{\sin((k+\frac{1}{2})x)}{\sin\frac{x}{2}} = \frac{1}{2N}\,\frac{1}{\sin^2\frac{x}{2}}\sum_{k=0}^{N-1}\sin((k+\tfrac{1}{2})x)\sin\frac{x}{2} =$$
$$= \frac{1}{2N\sin^2\frac{x}{2}}\sum_{n=0}^{N-1}\frac{1}{2}\left(\cos(nx) - \cos((n+1)x)\right) =$$
$$= \frac{1}{4N\sin^2\frac{x}{2}}\left(1 - \cos(Nx)\right) =$$
$$= \frac{1}{2N\sin^2\frac{x}{2}}\,\sin^2\left(\frac{N}{2}x\right)$$

□

Dokaz (▲):

$$\sigma_N(x) = \frac{1}{N}\left(S_0(x) + \dots + S_{N-1}(x)\right) =$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+y)\,F_N(y)\,dy = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,F_N(x-t)\,dt =$$

f je zvezna na $\mathbb{R}$ in 2π periodična $\Rightarrow$ enakomerno zvezna na $\mathbb{R}$
omejena na $\mathbb{R}$

$\sigma_N(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x+y) - f(x)\right)F_N(y)\,dy$; $F_N \ge 0$

$|\sigma_N(x) - f(x)| \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x+y) - f(x)|\,F_N(y)\,dy$

$\varepsilon > 0.\ \exists\delta > 0.\ \forall x:\ |y| \le \delta \Rightarrow |f(x+y) - f(x)| < \frac{\varepsilon}{2}$
(enakomerna zveznost)

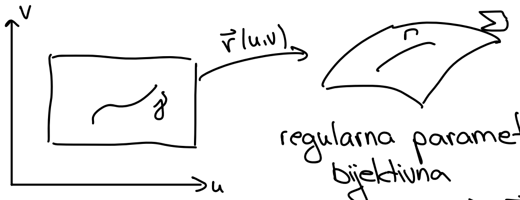
$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x+y)-f(x)|\,F_N(y)\,dy = \frac{1}{\pi}\int_{-\delta}^{\delta}|f(x+y)-f(x)|\,F_N(y)\,dy + \frac{1}{\pi}\int_{\delta<|y|\le\pi}|f(x+y)-f(x)|\,F_N(y)\,dy$$
$$\le \underbrace{\frac{1}{\pi}\int_{-\delta}^{\delta}\frac{\varepsilon}{2}F_N(y)\,dy}_{\le \frac{\varepsilon}{2}\cdot 1} + \frac{1}{\pi}\int_{\delta\le|y|\le\pi} 2\|f\|_\infty F_N(y)\,dy$$
$$\le \frac{\varepsilon}{2}\cdot 1 + \underbrace{\frac{2\|f\|_\infty}{\pi}\cdot 2\pi\cdot\max_{\delta\le|y|\le\pi}|F_N(y)|}_{\le\frac{\varepsilon}{2}} \le \varepsilon$$

$\le \frac{1}{2N}\frac{\pi^2}{\delta^2}$

$\Rightarrow N \ge N_0$

□

POVRŠINA PLOSKVE



regularna parametrizacija C^1
bijektivna
$\vec{r}_u \times \vec{r}_v \neq \vec{0}$

$$\vec{r}(u,v) = (X(u,v),\ Y(u,v),\ Z(u,v))$$

$$\text{rang}\begin{bmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{bmatrix} = 2$$

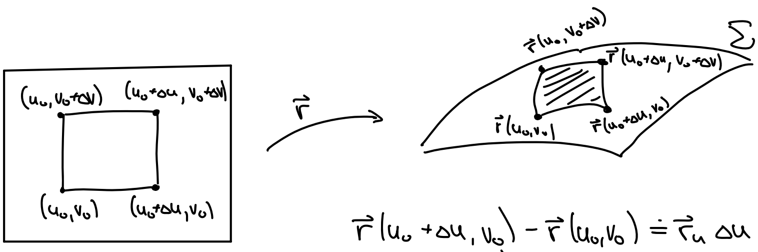
I fundamentalna forma: $\begin{bmatrix} E & F \\ F & G \end{bmatrix}(u,v) > 0$

$$E = \vec{r}_u \cdot \vec{r}_u \qquad F = \vec{r}_u \cdot \vec{r}_v \qquad G = \vec{r}_v \cdot \vec{r}_v$$

$$(\vec{r}_u \times \vec{r}_v)(\vec{r}_u \times \vec{r}_v) = |\vec{r}_u|^2 |\vec{r}_v|^2 = (\vec{r}_u \vec{r}_v)^2 > 0$$

Γ: $\vec{r}(u(t), v(t))$:
$$\ell(\Gamma) = \int_\alpha^\beta \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$

Iščemo formulo za površino Σ:

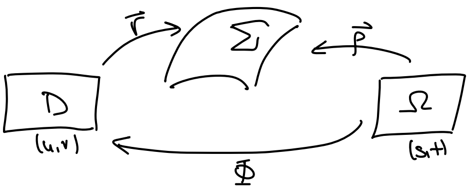


$$\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0) \doteq \vec{r}_u\, \Delta u$$
$$\vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0) \doteq \vec{r}_v\, \Delta v$$

$$P \approx |\vec{r}_u \times \vec{r}_v|\, \Delta u\, \Delta v$$

DEFINICIJA: Naj bo $D \overset{odp}{\subseteq} \mathbb{R}^2$ in $\vec{r}: D \to \Sigma$, $\Sigma \subseteq \mathbb{R}^3$, regularna parametrizacija ploskve Σ.
Potem je površina ploskve Σ:
$$P(\Sigma) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dv\, du = \iint_D \sqrt{EG - F^2}\, du\, dv$$

Naj bo:



$\vec{r}: D \to \Sigma$
$\vec{\rho}: \Omega \to \Sigma$ regularni parametrizaciji Σ

$\Phi: \Omega \to D$, $\Phi = \vec{r}^{-1} \circ \vec{\rho}$ difeomorfizem

Ali velja: $\iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv = \iint_\Omega |\vec{\rho}_s \times \vec{\rho}_t|\, ds\, dt$?

$$\underline{\Phi}(s,t) = (U(s,t), V(s,t))$$

$$\vec{r} \circ \underline{\Phi} = \vec{\rho}$$

$$\vec{r}(U(s,t), V(s,t)) = \vec{\rho}(s,t)$$

$$\vec{\rho}_s = \vec{r}_u \cdot U_s + \vec{r}_v V_s$$
$$\vec{\rho}_t = \vec{r}_u U_t + \vec{r}_v V_t$$

$$\vec{\rho}_s \times \vec{\rho}_t = (\vec{r}_u \times \vec{r}_v)(U_s V_t - U_t V_s)$$

$$|\vec{\rho}_s \times \vec{\rho}_t| = |\vec{r}_u \times \vec{r}_v| \cdot |J\underline{\Phi}|$$

upeljava novih spremenljivk

$$\iint_{\Omega} |\vec{\rho}_s \times \vec{\rho}_t| \, ds\, dt = \iint_{\Omega} |\vec{r}_u \times \vec{r}_v| |J\underline{\Phi}| \, ds\, dt = \iint_{D} |\vec{r}_u \times \vec{r}_v| \, du\, dv \qquad \square$$

POSLEDICA: $\Sigma = \mathrm{Graf}(f) = \{(x, y, f(x,y)) \, ; \, (x,y) \in D\}$
$f \in C^1(\bar{D})$

$$P(\Sigma) = P(\mathrm{Graf}(f)) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx\, dy$$

Dokaz:

$$\vec{r}(x,y) = (x, y, f(x,y))$$

$$\vec{r}_x = (1, 0, f_x) \qquad E = 1 + f_x^2$$
$$\vec{r}_y = (0, 1, f_y) \qquad F = f_x f_y$$
$$G = 1 + f_y^2$$

$$EG - F^2 = (1 + f_x^2)(1 + f_y^2) - (f_x f_y)^2$$

Zgled:

$$f(x,y) = \frac{1}{2}(x^2 + y^2) \; ; \; f_x = x, \; f_y = y$$

$D = K(0,1)$:

$$P = \iint_D \sqrt{1 + x^2 + y^2} \, dx\, dy = \int_0^{2\pi} \int_0^1 r \sqrt{1 + r^2} \, dr\, d\varphi = 2\pi (1 + r^2)^{\frac{3}{2}} \, \frac{2}{3} \cdot \frac{1}{2} \Big|_0^1 = \frac{2\pi}{3}(2^{\frac{3}{2}} - 1)$$

Zgled:

$S(0,a) \leftarrow$ sfera ; $a > 0$

$$\vec{r}(\varphi, \vartheta) = (a \cos\varphi \sin\vartheta, \; a \sin\varphi \sin\vartheta, \; a \cos\vartheta)$$
$$\varphi \in [0, 2\pi], \quad \vartheta \in [0, \pi]$$

$$\vec{r}_\varphi = (-a \sin\varphi \sin\vartheta, \; a \cos\varphi \sin\vartheta, \; 0)$$

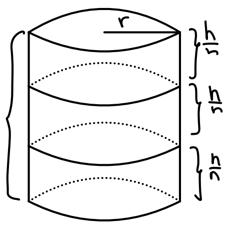
$$\vec{r}_\vartheta = (a \cos\varphi \cos\vartheta, \; a \cos\varphi \cos\vartheta, \; -a \sin\vartheta)$$

$$\left.\begin{aligned} E &= a^2\sin^2\vartheta \\ F &= 0 \\ G &= a^2 \end{aligned}\right\} \Rightarrow EG - F^2 = a^4 \sin^2\vartheta$$

$$E = |\vec{r_\varphi}|^2, \quad F = |r_\varphi r_\vartheta|, \quad G = |r_\vartheta|^2$$

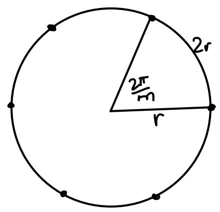
$$P = a^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta \, d\vartheta = 2\pi a^2 (-\cos\vartheta)\Big|_0^{\pi} = 4\pi a^2$$

SCHWARZOVA LANTERNA:

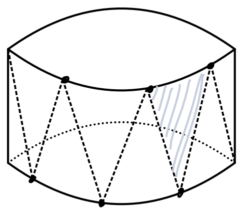


$P = 2\pi r h$

$n \in \mathbb{N}$

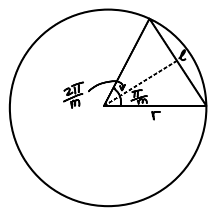


← m točk

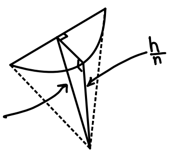


← 2m trikotnikov

vseh trikotnikov: 2mn



$l = 2r\sin\frac{\pi}{m}$



$v = \sqrt{\frac{h^2}{n^2} + 4r^2\sin^4\frac{\pi}{2m}}$

$$P = 2mn\, 2r \sin\frac{\pi}{m} \cdot \frac{1}{2} \sqrt{\frac{h^2}{n^2} + 4r^2\sin^4\frac{\pi}{2m}} = 2rmn \sin\frac{\pi}{m} \sqrt{\frac{h^2}{n^2} + 4r^2\sin^4\frac{\pi}{2m}} = P(m,n)$$

i) $m = n$:

$$P(m,m) = 2rm^2 \sin\frac{\pi}{m} \sqrt{\frac{h^2}{m^2} + 4r^2\sin^4\frac{\pi}{2m}} =$$

$$= 2r \frac{\sin\frac{\pi}{m}}{\frac{1}{m}} \sqrt{h^2 + 4r^2m^2\sin^4\frac{\pi}{2m}} \xrightarrow{m\to\infty} 2r\pi \sqrt{h^2 + 4r^2\left(\frac{\pi}{2}\right)^2 \cdot 0} = 2r\pi h$$

ii) $n = m^2$:

$$P(m,m^2) = 2rm^3 \sin\frac{\pi}{m} \sqrt{\frac{h^2}{m^4} + 4r^2\sin^4\frac{\pi}{2m}} =$$

$$= 2r \frac{\sin\frac{\pi}{m}}{\frac{1}{m}} \sqrt{h^2 + 4r^2 \frac{\sin^4\frac{\pi}{2m}}{\frac{1}{m^4}}} \xrightarrow{m\to\infty} 2r\pi \sqrt{h^2 + 4r^2\frac{\pi^4}{16}} > 2\pi r h$$

iii) $n = m^3$:

$$P(m,m^3) = 2rm^4 \sin\frac{\pi}{m} \sqrt{\frac{h^2}{m^6} + 4r^2\sin^4\frac{\pi}{2m}} =$$

$$= 2r \frac{\sin\frac{\pi}{m}}{\frac{1}{m}} \sqrt{h^2 + 4r^2m^4 \sin\frac{\pi}{2m}\, m^2} \xrightarrow{m\to\infty} \infty$$

5. VEKTORSKA ANALIZA

$\mathbb{R}^3$:

STANDARDNA ORTONORMIRANA BAZA: $\vec{e}_1, \vec{e}_2, \vec{e}_3 \sim \vec{i}, \vec{j}, \vec{k}$

neka druga ortonormirana baza: $\vec{p}, \vec{q}, \vec{r}$; $|\vec{p}| = |\vec{q}| = |\vec{r}| = 1$
$\vec{p} \cdot \vec{q} = \vec{p} \cdot \vec{r} = \vec{r} \cdot \vec{q} = 0$

Velja: $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$

Lahko je: $\vec{r} = \vec{p} \times \vec{q}$ ← POZITIVNO ORIENTIRANA ORTONORMIRANA BAZA

ali

$\vec{r} = -\vec{p} \times \vec{q}$ ← NEGATIVNO ORIENTIRANA ORTONORMIRANA BAZA

TERMINOLOGIJA:

$u : D^{\text{odp.}} \subseteq \mathbb{R}^3 \to \mathbb{R}$ ~ SKALARNO POLJE

$\vec{R} : D^{\text{odp.}} \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ ~ VEKTORSKO POLJE

Točka T: $\vec{e}_1, \vec{e}_2, \vec{e}_3$ $\quad$ $\vec{p}, \vec{q}, \vec{r}$
(x, y, z) $\quad$ (α, β, γ)
$$x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3 = \alpha\vec{p} + \beta\vec{q} + \gamma\vec{r}$$

Zgled:
$$\left.\begin{aligned} \vec{p} &= \tfrac{1}{\sqrt{2}}\vec{e}_1 + \tfrac{1}{\sqrt{2}}\vec{e}_3 \\ \vec{q} &= -\tfrac{1}{\sqrt{3}}\vec{e}_1 + \tfrac{1}{\sqrt{3}}\vec{e}_2 + \tfrac{1}{\sqrt{3}}\vec{e}_3 \\ \vec{r} &= \tfrac{1}{\sqrt{6}}\vec{e}_1 + \tfrac{2}{\sqrt{6}}\vec{e}_2 - \tfrac{1}{\sqrt{6}}\vec{e}_3 \end{aligned}\right\} \text{ ORTONORMIRANA BAZA}$$

$$\vec{p} = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right), \quad \vec{q} = \left(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \quad \vec{r} = \left(\tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right)$$

$$u(x,y,z) = x$$
$$\tilde{u}(\alpha,\beta,\gamma) = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \leftarrow \text{ortogonalna} \Rightarrow U^{-1} = U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \text{in} \quad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\tilde{u}(\alpha,\beta,\gamma) = u(U(\alpha,\beta,\gamma))$$

$$x = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$$
$$y = \frac{\beta}{\sqrt{3}} + \frac{2\gamma}{\sqrt{6}}$$
$$z = \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{3}} - \frac{\gamma}{\sqrt{6}}$$

$$\vec{R}(x,y,z) = \big(X(x,y,z),\ Y(x,y,z),\ Z(x,y,z)\big) \leftarrow \text{VEKTORSKO POLJE}$$

$$\tilde{\vec{R}}(\alpha,\beta,\gamma) = (U^T\vec{R} \circ U)(\alpha,\beta,\gamma) \qquad \tilde{u}(\alpha,\beta,\gamma) = (u \circ U)(\alpha,\beta,\gamma)$$

Zgled:

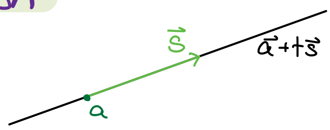
$$\vec{R}(x,y,z) = (x+2y+3z,\ x+2y+3z,\ x+2y+3z) = (x+2y+3z)\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$x+2y+3z = \frac{4\alpha}{\sqrt{2}} + \frac{4\beta}{\sqrt{3}} + \frac{2\gamma}{\sqrt{6}}$$

$$\tilde{\vec{R}}(\alpha,\beta,\gamma) = \left(\frac{4\alpha}{\sqrt{2}} + \frac{4\beta}{\sqrt{3}} + \frac{2\gamma}{\sqrt{6}}\right)\begin{bmatrix}\frac{2}{\sqrt{2}}\\ \frac{1}{\sqrt{3}}\\ \frac{2}{\sqrt{6}}\end{bmatrix}$$

SMERNI ODVOD SKALARNEGA POLJA

$u: D^{odp} \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$; $a \in D$, $\vec{s} \neq \vec{0}$



Smerni odvod skalarnega polja u v točki a v smeri vektorja $\vec{s}$ je:

$$\frac{\partial u}{\partial \vec{s}}(a) = \lim_{h\to 0} \frac{u(\vec{a}+h\vec{s}) - u(\vec{a})}{h} = \frac{d}{dt} u(\vec{a}+t\vec{s})$$

↖ če limita obstaja

$$\frac{\partial u}{\partial e_1} = \frac{\partial u}{\partial x} \qquad \frac{\partial u}{\partial e_2} = \frac{\partial u}{\partial y} \qquad \frac{\partial u}{\partial e_3} = \frac{\partial u}{\partial z}$$

Če je u diferenciabilna v a: $t \longmapsto u(\vec{a}+t\vec{s})$ je diferenciabilna v $t=0$

z odvodom: $(Du)(\vec{a})\cdot\vec{s}$

v bazi $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

$$(Du)(\vec{a}) = \left[\frac{\partial u}{\partial x}(\vec{a}) \quad \frac{\partial u}{\partial y}(\vec{a}) \quad \frac{\partial u}{\partial z}(\vec{a})\right]$$

V bazi $\vec{e}_1, \vec{e}_2, \vec{e}_3$: $\vec{s} = (s_1, s_2, s_3)$

$$(Du)(\vec{a})\vec{s} = \frac{\partial u}{\partial \vec{s}}(\vec{a}) = \frac{\partial u}{\partial x}(\vec{a})s_1 + \frac{\partial u}{\partial y}(\vec{a})s_2 + \frac{\partial u}{\partial z}(\vec{a})s_3 =$$

$$= \left(\frac{\partial u}{\partial x}(\vec{a}),\ \frac{\partial u}{\partial y}(\vec{a}),\ \frac{\partial u}{\partial z}(\vec{a})\right)\cdot\vec{s}$$

↖ skalarni produkt

u diferenciabilna na D:

$u \rightsquigarrow$ vektorsko polje na D.

GRADIENT u: $\operatorname{grad} u = \vec{\nabla} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = (u_x, u_y, u_z)$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

↖ NABLA

$$\vec{\nabla} u = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

Zgled:

$u(x,y,z) = x$

$\tilde{u}(\alpha,\beta,\gamma) = \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{6}}$

$\operatorname{grad} u = (1, 0, 0) \quad \leftarrow \vec{e}_1, \vec{e}_2, \vec{e}_3$

$\operatorname{grad} \tilde{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) \quad \leftarrow \vec{p}, \vec{q}, \vec{r}$

$$\operatorname{grad}\tilde{u} = U^T \operatorname{grad} u \circ U$$

$$\begin{bmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{2}{\sqrt6} & -\frac{1}{\sqrt6} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} \end{bmatrix}$$

$|\vec{s}| = 1$:
$$\frac{\partial u}{\partial \vec{s}} = \operatorname{grad} u \cdot \vec{s} = |\operatorname{grad} u| \cdot |\vec{s}| \cdot \cos\varphi = |\operatorname{grad} u| \cdot \cos\varphi$$

φ med $\vec{s}$ in $\operatorname{grad} u$

TRDITEV: Naj bo $u: D^{odp} \subseteq \mathbb{R}^3 \to \mathbb{R}$ skalarno polje, $a \in D$ in u diferenciabilna v a. Naj bo $\operatorname{grad} u(a) \neq \vec{0}$ in $|\vec{s}| = 1$. Smerni odvod u v točki a v smeri vektorja $\vec{s}$ je največji, če $\vec{s}$ kaže v smeri $\operatorname{grad} u$ in najmanjši, če $\vec{s}$ kaže v nasprotni smeri $\operatorname{grad} u$.

DIVERGENCA VEKTORSKEGA POLJA

vektorsko polje $\longrightarrow$ skalarno polje

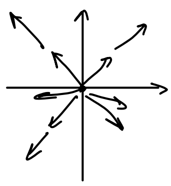
$C^1(D)$ $\quad$ $C(D)$

$\vec{R} \longrightarrow \operatorname{div}\vec{R}$

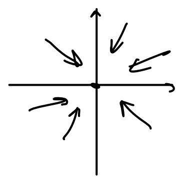
$(X, Y, Z)(x,y,z) \longrightarrow X_x + Y_y + Z_z = \operatorname{div}\vec{R}$

Zgled:

$\vec{R} = (X, Y, Z)$
$\operatorname{div}\vec{R} = 3$



$\vec{R} = (-X, -Y, -Z)$
$\operatorname{div}\vec{R} = -3$



meri končno ponorov in izvorov $\vec{R}$:

$$\operatorname{div}\vec{R} = \vec{\nabla}\vec{R}$$

Zgled:

$$\vec{R}(x,y,z) = (x+2y+3z,\ x+2y+3z,\ x+2y+3z)$$

$$\operatorname{div}\vec{R} = 1 + 2 + 3 = 6$$

$$U = \begin{bmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt6} \\ 0 & \frac{1}{\sqrt3} & \frac{2}{\sqrt6} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt3} & -\frac{1}{\sqrt6} \end{bmatrix} \qquad U^T = \begin{bmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{2}{\sqrt6} & -\frac{1}{\sqrt6} \end{bmatrix}$$

$$\vec{R}(\alpha,\beta,\gamma) = U^T \vec{R} \circ U$$

$$(\vec{R} \circ U)(\alpha,\beta,\gamma) = \left(\frac{4}{\sqrt2}\alpha + \frac{4}{\sqrt3}\beta + \frac{2}{\sqrt6}\gamma\right) \cdot (1,1,1)$$

$$x = \frac{\alpha}{\sqrt2} - \frac{\beta}{\sqrt3} + \frac{\gamma}{\sqrt6}$$
$$y = \frac{\beta}{\sqrt3} + \frac{2\gamma}{\sqrt6}$$
$$z = \frac{\alpha}{\sqrt2} + \frac{\beta}{\sqrt3} - \frac{\gamma}{\sqrt6}$$

$$\tilde{\vec{R}} = \left(\frac{4}{\sqrt{2}}\alpha + \frac{4}{\sqrt{3}}\beta + \frac{2}{\sqrt{6}}\gamma\right) U^T(1,1,1) = \left(\frac{4}{\sqrt{2}}\alpha + \frac{4}{\sqrt{3}}\beta + \frac{2}{\sqrt{6}}\gamma\right)\left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}\right) =$$

$$= \left(4\alpha + \frac{8}{\sqrt{6}}\beta + \frac{2}{\sqrt{3}}\gamma,\ \frac{4}{\sqrt{6}}\alpha + \frac{4}{3}\beta + \frac{2}{3\sqrt{2}}\gamma,\ \frac{4}{\sqrt{3}}\alpha + \frac{8}{3\sqrt{2}}\beta + \frac{2}{3}\gamma\right)$$

$$\operatorname{div}\tilde{\vec{R}} = \frac{8}{2} + \frac{4}{\sqrt{3}}\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}} = 6$$

ROTOR VEKTORSKEGA POLJA

$$\begin{array}{ccc} \text{vektorsko polje} & \longrightarrow & \text{vektorsko polje} \\ C^1(D) & & C(D) \\ \vec{R} & \longmapsto & \vec{\nabla}\times\vec{R} = \operatorname{rot}\vec{R} \end{array} \qquad \text{(curl)}$$

$$\vec{\nabla}\times\vec{R} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} = \underline{(Z_y - Y_z,\ X_z - Z_x,\ Y_x - X_y) = \operatorname{rot}\vec{R}}$$

Zgled: (nadaljevanje od prej)

$$\operatorname{rot}\vec{R} = (2-3,\ 3-1,\ 1-2) = (-1, 1, -1)$$

$$\operatorname{rot}\tilde{\vec{R}} = \left(\frac{8}{3\sqrt{2}} - \frac{2}{3\sqrt{2}},\ \frac{2}{\sqrt{3}} - \frac{4}{\sqrt{3}},\ \frac{4}{\sqrt{6}} - \frac{8}{\sqrt{6}}\right) = \left(\sqrt{2}, \frac{-2}{\sqrt{3}}, -\frac{4}{\sqrt{6}}\right)$$

$$U^T\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \left(-\sqrt{2}, \frac{2}{\sqrt{3}}, \frac{4}{\sqrt{6}}\right)$$

$\det U = -1 \Rightarrow$ baza $\vec{p}, \vec{q}, \vec{r}$ je negativno orientirana

Zgled:

$$\vec{R} = \vec{\omega}\times\vec{r} \qquad \vec{\omega} = (\omega_1, \omega_2, \omega_3),\quad \vec{r} = (x, y, z)$$

$$\vec{R} = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{pmatrix} = (\omega_2 z - \omega_3 y,\ \omega_3 x - \omega_1 z,\ \omega_1 y - \omega_2 x)$$

$$\operatorname{rot}\vec{R} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{pmatrix} = (2\omega_1, 2\omega_2, 2\omega_3) = 2\vec{\omega}$$

$\vec{R}$ meri koliko je vektorsko polje vrtinčasto

$$\begin{array}{ccc} \text{skalarno polje} & \to & \text{vektorsko polje} \\ u & \mapsto & \vec{\nabla}u = \operatorname{grad} u \end{array}$$

$$\begin{array}{ccc} \text{vektorsko polje} & \to & \text{skalarno polje} \\ \vec{R} & \mapsto & \vec{\nabla}\vec{R} = \operatorname{div}\vec{R} \end{array}$$

$$\begin{array}{ccc} \text{vektorsko polje} & \to & \text{vektorsko polje} \\ \vec{R} & \mapsto & \vec{\nabla}\times\vec{R} = \operatorname{rot}\vec{R} \end{array}$$

TRDITEV: Naj bo $D^{odp} \subseteq \mathbb{R}^3$.

1) Naj bo $u \in C^2(D)$ skalarno polje.
Tedaj:
$$\mathrm{rot}(\mathrm{grad}\, u) = 0$$
$$\vec{\nabla} \times \vec{\nabla} u = 0$$

2) Naj bo $\vec{R} \in C^2(D)$ vektorsko polje.
Tedaj je:
$$\mathrm{div}(\mathrm{rot}\, \vec{R}) = 0$$
$$\vec{\nabla}(\vec{\nabla} \times \vec{R}) = 0$$

Dokaz:

1) $\mathrm{grad}\, u = (u_x, u_y, u_z)$

enakost mešanih odvodov

$$\mathrm{rot}(\mathrm{grad}\, u) = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{pmatrix} = \left((u_z)_y - (u_y)_z,\ (u_x)_z - (u_z)_x,\ (u_y)_x - (u_x)_y\right) = \vec{0}$$

2) $\vec{R} = (X, Y, Z)$

$$\mathrm{rot}\, \vec{R} = (Z_y - Y_z,\ X_z - Z_x,\ Y_x - X_y)$$

$$\mathrm{div}(\mathrm{rot}\, \vec{R}) = (Z_y)_x - (Y_z)_x + (X_z)_y - (Z_x)_y + (Y_x)_z - (X_y)_z = 0$$

$\square$

$$u \rightsquigarrow \underset{\mathrm{grad}\, u}{(\vec{\nabla} u)} \rightsquigarrow \vec{\nabla}(\vec{\nabla} u) = \mathrm{div}(\mathrm{grad}\, u)$$

$$\Delta u = u_{xx} + u_{yy} + u_{zz}$$

$$\boxed{\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}} \leftarrow \text{LAPLACEOV OPERATOR}$$

VALOVNA ENAČBA: $u_{tt} = \Delta_{(x,y,z)} u$
$u_{tt} = u_{xx} + u_{yy} + u_{zz}$

TOPLOTNA ENAČBA: $u_t = \Delta_{(x,y,z)} u$

LAPLACEOVA ENAČBA: $\Delta u = 0$; $\Delta u = f$

DEFINICIJA: $u \in C^2(D)$ je HARMONIČNA na D, če je $\Delta u = 0$ na D.

Zgledi:

$u(x,y,z) = x^2 - y^2$
$u_{xx} = 2$
$u_{yy} = -2$

$u(x,y,z) = 2xy$

$\operatorname{rot}(\operatorname{grad} u) = 0$
$\operatorname{div}(\operatorname{rot}\vec{R}) = 0$

$\vec{R} \in C^1(D)$
Ali lahko sklepamo: $\operatorname{rot}\vec{R} = \vec{0} \Rightarrow \vec{R} = \operatorname{grad} u$ in
$\operatorname{div}\vec{R} = 0 \Rightarrow \vec{R} = \operatorname{rot}\vec{a}$? (V splošnem ne.)

Vektorsko polje $\vec{R}$ na $D^{odp} \subseteq \mathbb{R}^3$ je POTENCIALNO na D, če obstaja tak $u \in C^1(D)$, da je:
$$\vec{R} = \operatorname{grad} u$$
u je POTENCIAL $\vec{R}$.

$\operatorname{grad} u = \operatorname{grad} v \rightarrow \operatorname{grad}(u-v) = 0$
$\operatorname{grad} w = 0 \Leftrightarrow w_x = w_y = w_z = 0 \Leftrightarrow$ w konstantna funkcija na vsaki povezani komponenti D

D-povezana, vsi potenciali oblike $u_0 + C$

IROTACIONALNO vektorsko polje : $\operatorname{rot}\vec{R} = \vec{0}$
(brezvrtnično)

SOLENOIDNO vektorsko polje: $\operatorname{div}\vec{R} = 0$

Zgled:
$$\vec{R}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right) \qquad \mathbb{R}^3 \setminus \{(0,0,z);\ z \in \mathbb{R}\}$$
$$\operatorname{rot}\vec{R} = \left(0, 0, \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right) = \left(0, 0, \frac{(x^2+y^2) - x2x + (x^2+y^2) - 2y^2}{(x^2+y^2)^2}\right) = (0,0,0)$$

Ali obstaja $u \in C^1(\mathbb{R}^3 \setminus \{(0,0,z);\ z \in \mathbb{R}\})$, da je $\operatorname{grad} u = \vec{R}$?

$$u_x = \frac{-y}{x^2+y^2} \Rightarrow u(x,y,z) = \arctan\frac{y}{x} + A(y,z)$$
$$u_y = \frac{x}{x^2+y^2} \longrightarrow u_y = \frac{x}{x^2+y^2} = \frac{\frac{1}{x}x^2}{\left(1+\frac{y^2}{x^2}\right)x^2} + A_y \Rightarrow \left.\begin{array}{l} A_y = 0 \\ A_z = 0 \end{array}\right\} \Rightarrow A = A_0 \text{ konstanta}$$
$$u_z = 0$$

$$\Rightarrow u(x,y,z) = \arctan\frac{y}{x} + A_0$$
ne moremo dobro zvezno definirati na $\mathbb{R}^2 \setminus \{(0,0)\}$

$D^{odp} \subseteq \mathbb{R}^3$ je KONVEKSNA, če velja: $\forall \vec{a}, \vec{b} \in D \wedge \forall t \in [0,1] \Rightarrow t\vec{a} + (1-t)\vec{b} \in D$

$D^{odp} \subseteq \mathbb{R}^3$ je ZVEZDASTO OBMOČJE, če: $\exists a_0 \in D.\ \forall b \in D.\ \forall t \in [0,1]: ta_0 + (1-t)b \in D$

IZREK: Naj bo D zvezdasto območje v $\mathbb{R}^3$.

1) Naj bo $\vec{R} \in C^1(D)$ vektorsko polje.
Tedaj je $\vec{R}$ potencialno polje na $D \Leftrightarrow \operatorname{rot}\vec{R} = \vec{0}$.
$(\exists u \in C^2(D).\ \vec{R} = \operatorname{grad} u)$

2) Naj bo $\vec{R} \in C^1(D)$ vektorsko polje.
Potem ima $\vec{R}$ "vektorski potencial" $\Leftrightarrow \operatorname{div}\vec{R} = 0$
$(\exists \vec{G} \in C^2(D).\ \vec{R} = \operatorname{rot}\vec{G})$

Opombe: D zvezdasto polje:

1) $\operatorname{rot}\vec{R} = \vec{0}$: Koliko potencialov ima $\vec{R}$? Če je u_0 potencial $\vec{R}$ na D, so vsi potenciali $\vec{R}$ na D oblike $u = u_0 + C$
$\vec{R} = \operatorname{grad} u_0 = \operatorname{grad} u \Leftrightarrow \operatorname{grad}(u - u_0) = 0 \Leftrightarrow u = u_0 + C$

2) $\operatorname{div}\vec{R} = 0$: Koliko "vektorskih potencialov" ima $\vec{R}$?
$\vec{R} = \operatorname{rot}\vec{G}_0 = \operatorname{rot}\vec{G} \Rightarrow \operatorname{rot}(\vec{G} - \vec{G}_0) = \vec{0} \Leftrightarrow \vec{G} - \vec{G}_0 = \operatorname{grad} u$
$G = \vec{G}_0 + \operatorname{grad} u$

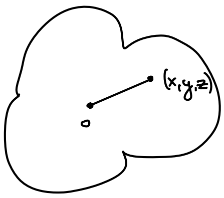
$\vec{G}$ vektorsko polje nad D:
$\operatorname{div}\vec{G} = f$ funkcija na D

DEJSTVO: $\exists u : D \to \mathbb{R}.\ \operatorname{div}(\operatorname{grad} u) = \Delta u = f$
$\operatorname{div}\vec{G} = \operatorname{div}(\operatorname{grad} u)$
$\operatorname{div}(\vec{G} - \operatorname{grad} u) = 0$

$$\boxed{\vec{G} = \operatorname{rot}\vec{F} + \operatorname{grad} u}$$

Dokaz (izrek):

(1) BŠS: D je zvezdasto glede na 0.



$(x,y,z) \in D$

$\forall t \in [0,1].\ (tx, ty, tz) \in D$

$\vec{R} = (X, Y, Z)(x,y,z)$

Definiramo: $u(x,y,z) = \int_0^1 [X(tx,ty,tz)x + Y(tx,ty,tz)y + Z(tx,ty,tz)z]\, dt$

Ali velja: $u_x = X$, $u_y = Y$, $u_z = Z$?

$\underline{u_x(x,y,z)} = \int_0^1 [X(tx,ty,tz) + tx X_x + ty Y_x + tz Z_x]\, dt = (*)$

$\vec{0} = \operatorname{rot}\vec{R} = (Z_y - Y_z,\ X_z - Z_x,\ Y_x - X_y)$

$(*) = \int_0^1 [X + tx X_x + ty X_y + tz X_z]\, dt = \int_0^1 \frac{d}{dt}(tX(tx,ty,tz))\, dt =$

$= \underline{X(x,y,z)},$

Podobno dobimo: $u_y = Y$, $u_z = Z$

Zgled: $\vec{R} = (y^2z^3+2,\ 2xyz^3+1,\ 3xy^2z^2)$

$$\mathrm{rot}\vec{R} = (6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3) = \vec{0}$$

$$u_x = y^2z^3+2 \Rightarrow u = xy^2z^2 + 2x + C(y,z)$$
$$u_y = 2xyz^3+1 = 2xyz^3 + C_y$$
$$u_z = 3xy^2z^2 \qquad \Rightarrow C_y = 1 \rightarrow C(y,z) = y + D(z)$$

$$\Rightarrow u = xy^2z^3 + 2x + y + Dz$$

$$\Rightarrow u_z = 3xyz^2 = 3xyz^2 + D' \quad \Rightarrow D = D_0$$

Zgled: $\vec{R} = (y^2z^3+2,\ 2xyz^3+x,\ 3xy^2z^3)$

$$u_x = y^2z^3+2 \Rightarrow u = xy^2z^3 + 2x + C(y,z)$$

$$u_y = 2xyz^3+x = 2xyz^3 + C_y(y,z)$$ tukaj bi morala biti odvisnost od x $\ast$

$$u_z = 3xy^2z^3$$

(2)

$\vec{R} = (X, Y, Z)$; $(x,y,z) \in D$

Definiramo:
$$\alpha(x,y,z) = \int_0^1 t X(tx, ty, tz)\, dt$$
$$\beta(x,y,z) = \int_0^1 t Y(tx, ty, tz)\, dt$$
$$\gamma(x,y,z) = \int_0^1 t Z(tx, ty, tz)\, dt$$

$\alpha, \beta, \gamma \in C^1(D)$

$$\mathrm{div}\vec{R} = 0 = X_x + Y_y + Z_z$$
$$\mathrm{div}(\alpha,\beta,\gamma) = \alpha_x + \beta_y + \gamma_z = \int_0^1 \left[t^2X_x + t^2Y_y + t^2Z_z\right] dt = 0$$

Definiramo: $\vec{G}(x,y,z) = (\alpha,\beta,\gamma) \times (x,y,z) = (x\beta - y\alpha,\ x\gamma - z\alpha,\ y\alpha - x\beta)$

rot$\vec{G} = \vec{R}$

$$\mathrm{rot}\,\vec{G} = (\alpha + y\alpha_y - x\beta_y - x\gamma_z + \alpha + z\alpha_z,$$

$$= (2\alpha + y\alpha_y + z\alpha_z - x\overbrace{(\beta_y + \gamma_z)}^{-\alpha_x},$$

$$= (2\alpha + x\alpha_x + y\alpha_y + z\alpha_z,$$

$$= \left(\int_0^1 \left[2tX + xt^2X_x + yt^2X_y + zt^2X_z\right] dt,\right.$$

$$= (X(x,y,z),\ Y(x,y,z),\ Z(x,y,z)) = \vec{R} \quad \square$$

MANJKA 2. in 3. KOORDINATA IZRAČUNAJ

Zgled:

$$\vec{R} = (2y-1,\ -1,\ 4x-2xy)$$

$$\operatorname{div}\vec{R} = 0+0+0 = 0$$

$$\alpha(x,y,z) = \int_0^1 t(2ty-1)\,dt = \int_0^1 2t^2y - t\,dt = \frac{2}{3}y - \frac{1}{2}$$

$$\beta(x,y,z) = \int_0^1 -t\,dt = -\frac{1}{2}$$

$$\gamma(x,y,z) = \int_0^1 t(4tx - 2t^2xy)\,dt = \frac{4}{3}x - \frac{2}{4}xy$$

$$G(x,y,z) = \begin{pmatrix} \frac{2}{3}y-\frac{1}{2} & -\frac{1}{2} & \frac{4}{3}x - \frac{1}{2}xy \\ x & y & z \end{pmatrix} =$$

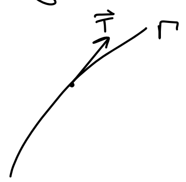
$$= \left(-\frac{1}{2}z - \frac{4}{3}xy + \frac{1}{2}xy^2,\ \frac{4}{3}x^2 - \frac{1}{2}x^2y - \frac{2}{3}zy + \frac{1}{2}z,\ \frac{3}{3}y^2 - \frac{1}{2}y + \frac{1}{2}x\right)$$

$$\underline{\operatorname{rot}\vec{G}} = \left(\frac{4}{3}y - \frac{1}{2} + \frac{2}{3}y - \frac{1}{2},\ -\frac{1}{2} - \frac{1}{2},\ \frac{8}{3}x - xy + \frac{4}{3}x - xy\right) = \underline{\vec{R}}$$

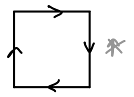
5.3. ORIENTABILNOST in ORIENTACIJA

KRIVULJE:

ORIENTACIJA GLADKE KRIVULJE je zvezen izbor enotskega tangentnega vektorja na krivuljo.



Če je gladka krivulja povezana, ima dve možni različni orientaciji $\vec{T}$ in $-\vec{T}$.

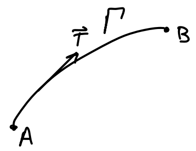


KRIVULJE Z ROBOM: (povezane)

model: zaprt interval $[0,1]$

$$\Gamma = \vec{r}([0,1])$$

robni točki $\vec{r}(0), \vec{r}(1): A, B$

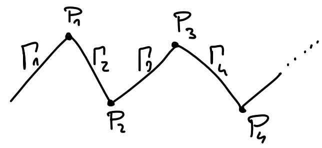


Orientacija Γ porodi orientacijo roba Γ: $\{A, B\}$
Ena od teh dveh točk je začetna točka, druga pa je končna točka.

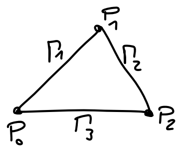
ODSEKOMA GLADKE KRIVULJE:

$\Gamma = \Gamma_1 \cup \dots \cup \Gamma_n$; $\Gamma_1, \dots, \Gamma_n$ gladke krivulje

$$\Gamma_1 \cap \Gamma_2 = \{P_1\}$$
$$\Gamma_2 \cap \Gamma_3 = \{P_2\}$$
$$\vdots$$
$$\Gamma_{n-1} \cap \Gamma_n = \{P_{n-1}\}$$



ostalih presečišč ni, razen morda $\Gamma_n \cap \Gamma_1 = \{P_0\}$



Orientacija Γ je podana z orientacijami $\Gamma_1, \dots, \Gamma_n$, ki na presečiščih inducirajo nasprotno orientacijo.

Vsaka krivulja je ORIENTABILNA.

ORIENTACIJA GLADKE PLOSKVE:

ORIENTACIJA GLADKE PLOSKVE je zvezen izbor enotske normale ploskve. (če obstaja)

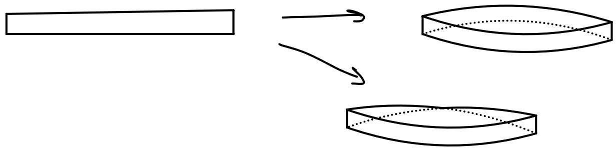
$\vec{r}: D \subseteq \mathbb{R}^2 \longrightarrow \Sigma \subseteq \mathbb{R}^3$ regularna parametrizacija

$\vec{N} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ ← zvezna enotska normala na Σ

Če na Σ lahko izberemo orientacijo, je ORIENTABILNA ploskev.

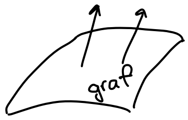
Če je Σ orientabilna in povezana, ima dve možni orientaciji: $-\vec{N}$ in $\vec{N}$.

MÖBIUSOV TRAK:



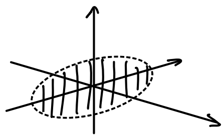
Graf:

$\Sigma = \{(x,y,f(x,y)), (x,y) \in D\}$ $\Rightarrow \vec{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$ ← $u(x,y,z) = z - f(x,y)$
↑ gradient

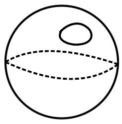


PLOSKVE Z ROBOM:

$\partial\Gamma = \{A, B\}$

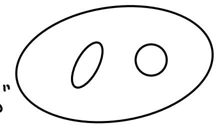


$\partial\Sigma = \overline{\Sigma} \setminus \Sigma$ = končno mnogo odsekoma gladkih krivulj



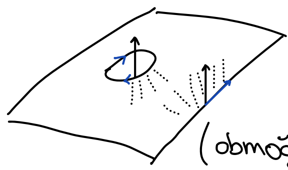
"sfera z luknjo" (topološko enako disku)

"ploskev z luknjo"



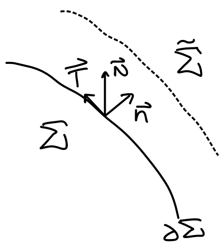
Σ orientirana ploskev z robom

KOHERENTNA oz. USKLAJENA orientacija $\partial\Sigma$



(območje ploskve na levi strani)

Robne krivulje so orientirane skladno z orientacijo $(\Sigma, \vec{N})$ tako, da če stojimo v robni točki z glavo v smeri normale $\vec{N}$ in gledamo v smeri orientacije $\partial\Sigma$, potem je Σ na naši levi strani.



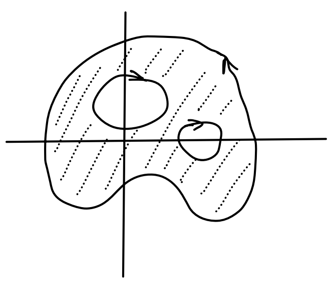
$\overline{\Sigma} \subseteq \tilde{\Sigma}$ ploskev

$\vec{n}$ ... zunanja normala

$\vec{T} = \vec{N} \times \vec{n}$
↑ orientacija roba

$\Sigma \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^2 \times \{0\}$:

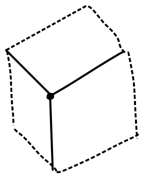
STANDARDNA ORIENTACIJA $\Sigma \subseteq \mathbb{R}^2$ je $\vec{N} = (0,0,1)$



ODSEKOMA GLADKE PLOSKVE:

$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$; kjer Σ_i gladke ploskve z robom $\partial\Sigma_i$

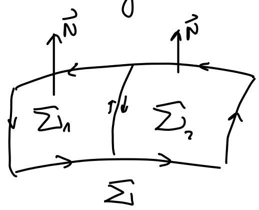
Ali velja $\overline{\Sigma}_j \cap \overline{\Sigma}_k = \emptyset$, $j \neq k$, oziroma $\overline{\Sigma}_j \cap \overline{\Sigma}_k \subseteq \partial\Sigma_j \cap \partial\Sigma_k$

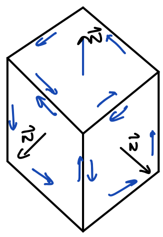


j, k, ℓ ; $j \neq k$, $j \neq \ell$, $k \neq \ell$: $\partial\Sigma_j \cap \partial\Sigma_k \cap \partial\Sigma_\ell = \begin{cases} \emptyset \\ \text{končna množica točk} \end{cases}$

Presek treh ali več robov $\partial\Sigma_i$ je končna množica točk (ni krivulja)

ORIENTACIJA ODSEKOMA GLADKE PLOSKVE $\Sigma = \bigcup \Sigma_i$ ($(\Sigma_i, \vec{N}_i)$ orientacija Σ_i) je podana z orientacijami $(\Sigma_i, \vec{N}_i)$, če so na delih robov $\partial\Sigma_i$, ki se sekajo vzdolž krivulj, koherentne orientacije robov nasprotne.





5.2/5.4 KRIVULJNI in PLOSKOVNI INTEGRALI

Dva tipa integralov:

1) INTEGRAL SKALARNEGA POLJA PO KRIVULJI OZ. PLOSKVI
(orientacija ni pomembna)

2) INTEGRAL VEKTORSKEGA POLJA PO KRIVULJI OZ. PLOSKVI
(orientacija je pomembna)

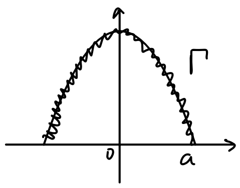
INTEGRAL SKALARNEGA POLJA PO KRIVULJI:

$\Gamma \subseteq \mathbb{R}^3$ (gladka) krivulja in $u: \Gamma \to \mathbb{R}$ zvezno skalarno polje
$\vec{r}: [\alpha, \beta] \to \Gamma$ regularna parametrizacija

$$\int_\Gamma u\,ds = \int_\alpha^\beta u(\vec{r}(t)) \cdot |\dot{\vec{r}}(t)|\,dt$$

- $u \equiv 1 \Rightarrow$ dolžina Γ
- kot pri dolžini se pokaže, da je vrednost neodvisna od regularne parametrizacije

Zgled:



homogena polkrožnica ($\rho = \rho_0$) ← dolžinska gostota

$$m = \frac{1}{2} 2\pi a \rho_0 = \pi a \rho_0$$

$x = a\cos\varphi$ $\quad \varphi \in [0, \pi]$
$y = a\sin\varphi$ $\quad a > 0$ $\quad \Rightarrow \dot{x} = -a\sin\varphi$
$\dot{y} = a\cos\varphi \qquad \sqrt{\dot{x}^2 + \dot{y}^2} = a = |\dot{\vec{r}}|$

$x_T = 0$

$$\Rightarrow \underline{T} = \frac{1}{m}\int_\Gamma y\rho_0\,ds = \frac{1}{\pi a \rho_0}\int_0^\pi \rho_0 a^2 \sin\varphi\,d\varphi = \frac{a}{\pi}\int_0^\pi \sin\varphi\,d\varphi = \underline{\frac{2}{\pi}a},$$

$$m = \int_\Gamma \rho_0\,ds = \rho_0 \int_0^\pi a\,d\varphi = \pi a \rho_0$$

INTEGRAL VEKTORSKEGA POLJA PO ORIENTIRANI KRIVULJI:

$\vec{\Gamma} = (\Gamma, \vec{T})$; $\quad \vec{R}: \Gamma \to \mathbb{R}^3$ zvezno vektorsko polje $\qquad -\vec{\Gamma} = (\Gamma, -\vec{T})$

$$\int_{\vec{\Gamma}} \vec{R}\,d\vec{r} = \int_\Gamma \vec{R} \cdot \vec{T}\,dS$$

<u>Opomba</u>: Če na Γ izberemo nasprotno orientacijo, se predznak rezultata spremeni:

$$\int_{-\vec{\Gamma}} \vec{R}\,d\vec{r} = -\int_{\vec{\Gamma}} \vec{R}\,d\vec{r}$$

Če je $\vec{r}: [\alpha, \beta] \to \Gamma$ regularna parametrizacija, usklajena z orientacijo: $\frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \vec{T}$:

$$\int_{\vec{\Gamma}} \vec{R}\,d\vec{r} = \int_\alpha^\beta \vec{R}(\vec{r}(t)) \cdot \dot{\vec{r}}(t)\,dt$$

Če je Γ odsekoma gladka: $\Gamma = \bigcup_i \Gamma_i$: $\int_\Gamma \vec{R}\,d\vec{r} = \sum_{j=1}^{n} \int_{\Gamma_j} \vec{R}\,d\vec{r}$

Zgled:

$\Gamma = \{(t, t^2, t^3);\ t \in [0,1]\}$ $\quad \vec{r}(t) = (t, t^2, t^3)$

$\dot{\vec{r}}(t) = (1, 2t, 3t^2) \rightsquigarrow \vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$ orientacija

$\vec{R} = (x, y, z) = \vec{\nabla}\left(\frac{1}{2}(x^2+y^2+z^2)\right)$ potencialno

$$\int_\Gamma \vec{R}\,d\vec{r} = \int_0^1 (t, t^2, t^3)\cdot(1, 2t, 3t^2)\,dt = \int_0^1 t + 2t^3 + 3t^5\,dt = \frac{1}{2} + \frac{2}{4} + \frac{3}{6} = \frac{3}{2}$$

$$(= u(1,1,1) - u(0,0,0))$$

INTEGRAL SKALARNEGA POLJA PO PLOSKVI:

$\Sigma \subseteq \mathbb{R}^3$ ploskev; $u: \Sigma \to \mathbb{R}$ zvezno skalarno polje; $\vec{r}: D \subseteq \mathbb{R}^2_{(u,v)} \to \Sigma$ regularna parametrizacija

$$\iint_\Sigma u\,dS := \iint_D u(\vec{r}(s,t)) \cdot \sqrt{EG - F^2}\,ds\,dt \qquad \begin{aligned} E &= |\vec{r}_s|^2 \\ F &= \vec{r}_s \cdot \vec{r}_t \\ G &= |\vec{r}_t|^2 \end{aligned}$$

$$= \iint_D u(\vec{r}(s,t))\,|\vec{r}_s \times \vec{r}_t|\,ds\,dt$$

Kot pri površini $(u=1)$ se izkaže, da je definicija neodvisna od regularne parametrizacije.

Če je $\Sigma = \bigcup \Sigma_i$ odsekoma gladka ploskev, potem je:

$$\iint_\Sigma u\,dS = \sum_{i=1}^{n} \iint_{\Sigma_i} u\,dS$$

Zgled: težišče homogene polsfere

$a > 0$; $\rho = \rho_0$ površinska gostota

$$m = \frac{1}{2} 4\pi a^2 \rho_0 = 2\pi a^2 \rho_0$$

$x_T = y_T = 0$

$$z_T = \frac{1}{m} \iint_\Sigma z \rho_0\,dS = \frac{1}{2\pi a^2} \iint_\Sigma z\,dS = (*)$$

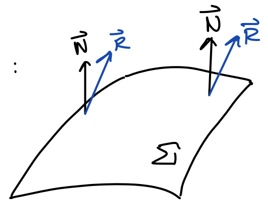
Parametrizacija: $\begin{aligned} x &= a\cos\varphi \sin\vartheta \\ y &= a \sin\varphi \sin\vartheta \\ z &= a\cos\vartheta \end{aligned}$ $\quad \varphi \in [0, 2\pi],\ \vartheta \in [0, \frac{\pi}{2}]$

$\Rightarrow \sqrt{EG - F^2} = a^2 \sin\vartheta$

$$(*) = \frac{1}{2\pi a^2} \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} a^3 \cos\vartheta \cdot \sin\vartheta\,d\vartheta = \frac{a}{2\pi} \cdot 2\pi \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin(2\vartheta)\,d\vartheta =$$

$$= \frac{a}{2}$$

INTEGRAL VEKTORSKEGA POLJA PO ORIENTIRANI PLOSKVI:



$\vec{\Sigma} = (\Sigma, \vec{N})$; $\vec{R} : \Sigma \to \mathbb{R}^3$ zvezno vektorsko polje

$$\iint_{\vec{\Sigma}} \vec{R}\, d\vec{S} := \iint_{\Sigma} \vec{R}\cdot\vec{N}\; dS$$

<u>Opomba:</u> Če orientacijo obrnemo, se vrednost integrala spremeni za predznak.

Če je $\vec{r} : D_{(s,t)} \to \Sigma$ regularna parametrizacija, usklajena z orientacijo, potem je: $\vec{N} = \frac{\vec{r}_s \times \vec{r}_t}{|\vec{r}_s \times \vec{r}_t|}$ in

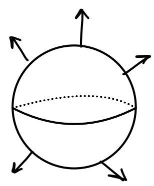
$$\iint_{\vec{\Sigma}} \vec{R}\, d\vec{S} = \iint_D \vec{R}(\vec{r}(s,t))\cdot(\vec{r}_s(s,t), \vec{r}_t(s,t))\; ds\,dt$$

$$= \iint_D [\vec{R}(\vec{r}), \vec{r}_s, \vec{r}_t](s,t)\; ds\,dt$$

Zgled:

$\vec{\Sigma} = S(0, R_0)$, $R_0 > 0$

$\vec{N} = (0,0,R_0) = (0,0,1)$



$\vec{R} = \left(\frac{x}{w^2}, \frac{y}{w^2}, \frac{z}{w^2}\right)$, kjer $w = x^2+y^2+z^2$

$$\iint_{S(0,R_0)} \vec{R}\, d\vec{S} = \iint_{S(0,R_0)} \frac{1}{R_0^3}(x,y,z)\, d\vec{S} = \qquad \vec{N} = \frac{(x,y,z)}{R_0}$$

$$= \iint_{S(0,R_0)} \frac{1}{R_0^3}\,\frac{R_0^2}{R_0}\, dS = \frac{1}{R_0^2}\iint_{S(0,R_0)} 1\, dS = \frac{4\pi R_0^2}{R_0^2} = 4\pi$$

KRIVULJNI INTEGRAL POTENCIALNEGA POLJA

$\vec{R} = (u_x, u_y, u_z)$; u potencial, $u \in C^1(D)$

$\vec{r}(t)$ parametrizacija

$\vec{r} : [\alpha, \beta] \to \Gamma$

$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$

$\vec{r}(t) = (x(t), y(t), z(t))$

$\vec{\Gamma} \subseteq D$:

$$\int_{\vec{\Gamma}} \vec{R}\, d\vec{r} = \int_{\Gamma} \vec{\nabla} u\, d\vec{r} =$$

$$= \int_\alpha^\beta (u_x, u_y, u_z)(\vec{r}(t))\cdot\dot{\vec{r}}(t)\, dt =$$

$$= \int_\alpha^\beta (u_x\dot{x} + u_y\dot{y} + u_z\dot{z})(t)\; dt =$$

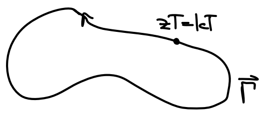
$$= \int_\alpha^\beta \frac{d}{dt}\left(u(\vec{r}(t))\right) dt =$$

$$= u(\vec{r}(\beta)) - u(\vec{r}(\alpha))$$

TRDITEV: Integral potencialnega vektorskega polja po orientirani krivulji je enak razliki potenciala med končno in začetno točko krivulje.

Sklenjena orientirana krivulja: $\vec{\Gamma}$ in $zT = kT$ (začetna točka = končna točka)

$$\oint_{\vec{\Gamma}} \vec{R}\,d\vec{r}$$



TRDITEV: Naj bo $\vec{R}: D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ zvezno vektorsko polje.
Naslednje trditve so ekvivalentne:

(1) $\vec{R}$ je potencialno

(2) Integral $\vec{R}$ po orientirani krivulji med poljubnima dvema točkama A in B v D (A začetna, B končna za $\vec{\Gamma}$) je neodvisen od poti oz. krivulje od A do B.

$$\forall A,B \in D.\ \forall \vec{\Gamma}_1, \vec{\Gamma}_2 \subseteq D.\ A \text{ začetna } \vec{\Gamma}_1, \vec{\Gamma}_2,\ B \text{ končna } \vec{\Gamma}_1, \vec{\Gamma}_2 :\ \int_{\vec{\Gamma}_1} \vec{R}\,d\vec{r} = \int_{\vec{\Gamma}_2} \vec{R}\,d\vec{r}$$

(3) Naj bo $\vec{\Gamma}$ sklenjena pot oz. krivulja v D. Tedaj: $\oint_{\vec{\Gamma}} \vec{R}\,d\vec{r} = 0$.

Zgled:

$$\vec{R} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right), \quad \operatorname{rot}\vec{R} = \vec{0}$$

$\vec{\Gamma} = K(0, r_0),\ r_0 > 0$ ← krožnica

$x = r_0 \cos t$
$y = r_0 \sin t$ $\qquad \dot{\vec{r}} = (-r_0 \sin t, r_0 \cos t, 0)$
$z = 0$

$$\vec{R} = (r_0 \cos t, r_0 \sin t, 0) = \left(\frac{-r_0 \sin t}{r_0^2}, \frac{r_0 \cos t}{r_0^2}, 0\right)$$

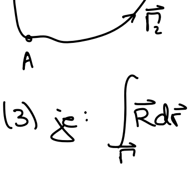
$$\oint_{\vec{\Gamma}} \vec{R}\,d\vec{r} = \int_0^{2\pi} d\varphi = 2\pi \neq 0$$

Dokaz:

$(1) \Rightarrow (3)$: $\vec{R} = \vec{\nabla} u$ potencialno

$$\oint_{\vec{\Gamma}} \vec{R}\,d\vec{r} = u(kT) - u(zT) = 0, \text{ saj je } \vec{\Gamma} \text{ sklenjena}$$

$(3) \Rightarrow (2)$:



$$\vec{\Gamma} = \vec{\Gamma}_1 \cup (-\vec{\Gamma}_2)$$

Po (3) je: $$\int_{\vec{\Gamma}} \vec{R}\,d\vec{r} = 0 = \int_{\vec{\Gamma}_1} \vec{R}\,d\vec{r} + \int_{-\vec{\Gamma}_2} \vec{R}\,d\vec{r} = \int_{\vec{\Gamma}_1} \vec{R}\,d\vec{r} - \int_{\vec{\Gamma}_2} \vec{R}\,d\vec{r} = 0$$

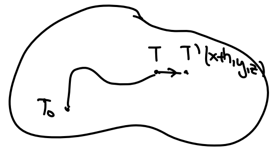
$(2) \Rightarrow (1)$: D povezana $\Rightarrow$ povezana s potmi

$T_0 \in D$, $T \in D$: Izberemo $\vec{\Gamma}$ ... krivulja med T_0 in T.

Definiramo: $u(T) = \int_{\vec{\Gamma}} \vec{R}\,d\vec{r}$ (dobro definirano zaradi (2))

Ali velja $\vec{\nabla} u = \vec{R}$? $\vec{R} = (X, Y, Z)$ $\qquad (u_x = X,\ u_y = Y,\ u_z = Z\ ?)$

$$u_x(x,y,z) = \lim_{h \to 0} \frac{u(x+h,y,z) - u(x,y,z)}{h} =$$



$$= \lim_{h\to 0} \int_{|(x,y,z)(x+th,y,z)|} \vec{R}\,d\vec{r} =$$

$\leftarrow$ daljica $t \mapsto (x+th, y, z)$

$$= \lim_{h\to 0} \frac{1}{h} \int_0^1 (X, Y, Z)(h, 0, 0)\, dt =$$

$$= \lim_{h\to 0} \int_0^1 X(x+th, y, z)\, dt =$$

$$= X(x, y, z)$$

Podobno dokažemo $u_y = Y$ in $u_z = Z$. $\square$

5.5. INTEGRALSKI IZREKI

$$a<b: \int_a^b df(x) = \int_a^b f'(x)\,dx = f(b)-f(a)$$

$$\int_a^b f'(x)\,dx = \int_{[a,b]} f\,dx$$

$$\int_a^b f'(x)\,dx = f(b)-f(a) = \int_{\partial[a,b]} f(x)\,dx \leftarrow \text{orientiran rob}$$

$$\partial[a,b] = \{a,b\}$$

ZT (pri a), KT (pri b)

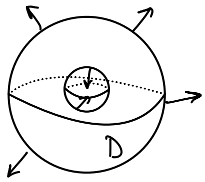
Omejena mnogoterost: M orientirana, ∂M skladno orientiran

$d\omega$ "odvod", ω vektorsko polje

$$\int_M d\omega = \int_{\partial M} \omega \qquad \text{(STOKESOV IZREK)}$$

GAUSSOV IZREK

Naj bo D omejena odprta množica v $\mathbb{R}^3$, katere rob je sestavljen iz končnega števila odsekoma gladkih ploskev, orientiranih z zunanjo normalo glede na D.



Naj bo $\vec{R} \in C^1(\bar{D})$ vektorsko polje.
Tedaj:
$$\iint_{\partial D} \vec{R}\,d\vec{S} = \iiint_D \operatorname{div}\vec{R}\,dV.$$

Zgled:

$D = K(0,R_0) \subseteq \mathbb{R}^3$, $R_0>0$; $\vec{R} = (x,y,z)$ in $\operatorname{div}\vec{R} = 3$

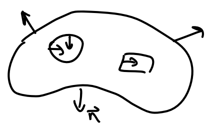
$$\iiint_{K(0,R_0)} 3\,dV = 3\,\frac{4\pi R_0^3}{3} = 4\pi R_0^3$$

$$\vec{N} = \frac{(x,y,z)}{R_0}$$

$$\iint_{\partial K(0,R_0)} \vec{R}\,d\vec{S} = \iint_{\partial K(0,R_0)} (x,y,x)\,\frac{(x,y,z)}{R_0}\,dS = R_0 \iint_{\partial K(0,R_0)} 1\,dS = 4\pi R_0^3$$

GAUSSOV IZREK V $\mathbb{R}^2$:

D... omejena odprta množica v $\mathbb{R}^2$
∂D... končna unija odsekoma gladkih krivulj
$\vec{n}$ ... enotska zunanja normala
$\vec{R} = (M,N)$ vektorsko polje $C^1(\bar{D})$



$$\int_{\partial D} \vec{R}\vec{n}\,ds = \iint_D \operatorname{div}\vec{R}\,dS = \iint_D (M_x + N_y)\,dx\,dy$$

$\overline{D} = \overline{D}_1 \cup \overline{D}_2 \subseteq \mathbb{R}^3$. Če Gaussov izrek velja za D_1 in D_2, potem velja tudi za D.

Naj velja Gaussov izrek za $\vec{R}$ na D_1 in $\vec{R}$ na D_2:

$$\iint_{\partial D_1} \vec{R}\, d\vec{S} = \iiint_{D_1} \operatorname{div}\vec{R}\, dV \quad \text{in} \quad \iint_{D_2} \vec{R}\, d\vec{S} = \iiint_{D_2} \operatorname{div}\vec{R}\, dV.$$

$$\iiint_{\overline{D}_1} \operatorname{div}\vec{R}\, dV + \iiint_{\overline{D}_2} \operatorname{div}\vec{R}\, dV = \iiint_{\overline{D} = \overline{D}_1 \cup \overline{D}_2} \operatorname{div}\vec{R}\, dV$$

(zunanji normali D_1 in D_2 na "preseku" sta nasprotni $\rightarrow \int$ na tistem delu $= 0$)

Vsota integralov po skupnem robu D_1 in D_2 je zaradi orientacije enaka 0.
Zato je:

$$\iint_{\partial D_1} \vec{R}\, d\vec{S} + \iint_{\partial D_2} \vec{R}\, d\vec{S} = \iint_{\partial D} \vec{R}\, d\vec{S}$$

Torej Gaussov izrek velja tudi za $\vec{R}$ na D.

GREENOVA FORMULA

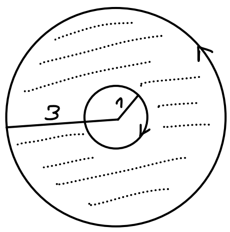
Naj bo $D \subseteq \mathbb{R}^2$ omejena odprta podmnožica ravnine, katere rob je sestavljen iz končnega števila odsekoma gladkih krivulj, orientiranih pozitivno glede na D.
Naj bosta $X, Y \in C^1(\bar{D})$.
Tedaj:

$$\int_{\partial D} X dx + Y dy = \iint_D (Y_x - X_y) dx dy = \iint_D \operatorname{rot}\vec{R} \cdot \vec{N} \, dS$$

Komentar: $\vec{R}(x,y,z) = (X(x,y), Y(x,y), 0)$: $\operatorname{rot}\vec{R} = (0, 0, Y_x - X_y)$, $\vec{N} = (0,0,1)$

Zgled:

$D = \{(x,y,z) \in \mathbb{R}^2 ;\ 1 < x^2+y^2 < 9\}$



$X(x,y) = -y, \quad Y(x,y) = x$

$$\iint_D (Y_x - X_y) dx dy = \iint_D (1+1) dx dy = 2P(D) = 2(9\pi - 1\pi) = 16\pi$$

$$\int_{\partial D} X dx + Y dy = \underbrace{\int_{\sqrt{x^2+y^2}=3} X dx + Y dy}_{I} + \underbrace{\int_{\sqrt{x^2+y^2}=1} X dx + Y dy}_{II} = 18\pi - 2\pi = 16\pi$$

$x = 3\cos t \qquad dx = -3\sin t \, dt$

$y = 3\sin t \qquad dy = 3\cos t \, dt$

$$I = \int_0^{2\pi} \underbrace{(-3\sin t)}_{-y}(-3\sin t) dt + \int_0^{2\pi} \underbrace{(3\cos t)}_{x}(3\cos t) dt = 9 \cdot 2\pi = 18\pi$$

$x = \cos t \qquad t \in [2\pi, 0]$

$y = \sin t$

$$II = \int_{2\pi}^{0} (-\sin t)(-\sin t) dt + \int_{2\pi}^{0} \cos t \cdot \cos t \, dt = -2\pi$$

Gaussov izrek v $\mathbb{R}^2 \Rightarrow$ Greenova formula

$X, Y \in C^1(\bar{D})$ $\qquad \vec{n} = (N_1, N_2) \to \vec{T} = (-N_2, N_1)$

$$\int_{\partial D} X dx + Y dy = \iint_D (Y_x - X_y) dx dy$$

Gauss

$$\int_{\partial D} (X,Y) d\vec{r} = \int_{\partial D} (X,Y) \cdot \vec{T} ds = \int_{\partial D} (X,Y)(-N_2, N_1) ds = \int_{\partial D} (YN_1 + (-X)N_2) ds = \int_{\partial D} (Y, -X)\vec{n} ds =$$

$$= \iint_D (Y_x - X_y) dx dy$$

STOKESOV IZREK

Naj bo Σ omejena, odsekoma gladka orientirana ploskev v $\mathbb{R}^3$, katere rob je sestavljen iz končnega števila odsekoma gladkih krivulj, orientiranih skladno z orientacijo Σ.
Naj bo $\vec{R} \in C^1(\overline{\Sigma})$ vektorsko polje.
Tedaj:
$$\int_{\partial\Sigma} \vec{R}\,d\vec{r} = \iint_{\Sigma} \operatorname{rot}\vec{R}\,d\vec{S}.$$

Komentar: $\partial\Sigma = \emptyset \Rightarrow \iint_{\Sigma} \operatorname{rot}\vec{R}\,d\vec{S} = 0$

Zgled:

$$\vec{R} = \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) = \operatorname{rot}\vec{G}$$
ne obstaja

$\mathbb{R}^3 \setminus \{(0,0,0)\}$

$$\operatorname{div}\vec{R} = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + x\left(-\frac{3}{2}\right)\frac{2x}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + y\left(-\frac{3}{2}\right)\frac{2y}{(x^2+y^2+z^2)^{\frac{5}{2}}} +$$
$$+ \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + z\left(-\frac{3}{2}\right)\frac{2z}{(x^2+y^2+z^2)^{\frac{5}{2}}} =$$
$$= 0$$

Zgled:

$R_0 > 0$, $\Sigma = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 = R_0^2;\ z > 0\}$
$\vec{N}(0,0,R_0) = (0,0,1)$

zgornja hemisfera

$$\vec{R} = (zy, x, x-y+z)$$

$$\operatorname{rot}\vec{R} = (-1, y-1, 1-z) \rightarrow \operatorname{div}(\operatorname{rot}\vec{R}) = 0$$

$$\vec{N} = \frac{(x,y,z)}{R_0}$$

$$\underline{\iint_{\Sigma} \operatorname{rot}\vec{R}\,d\vec{S}} = \iint_{\Sigma} (-1, y-1, 1-z)\cdot\vec{N}\,dS = \frac{1}{R_0}\iint_{\Sigma} (-1, y-1, 1-z)(x,y,z)\,dS =$$

$$= \frac{1}{R_0}\iint_{\Sigma} (-x + y^2 - y + z - z^2)\,dS =$$

$x = R_0 \cos\varphi \sin\vartheta$, $\quad \varphi \in [0, 2\pi]$
$y = R_0 \sin\varphi \sin\vartheta$, $\quad \vartheta \in [0, \frac{\pi}{2}]$
$z = R_0 \cos\vartheta$, $\quad dS = R_0^2 \sin\vartheta\,d\varphi\,d\vartheta$

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \left(-\overset{0}{\cos\varphi}\sin\vartheta + R_0\sin^2\varphi\sin^2\vartheta - \overset{0}{\sin\varphi}\sin\vartheta + \cos\vartheta - R_0\cos^2\vartheta\right)R_0^2\sin\vartheta\,d\vartheta =$$

$$= R_0^2 \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \left(R_0\sin^2\varphi\sin^3\vartheta + \cos\vartheta\sin\vartheta - R_0\cos^2\vartheta\sin\vartheta\right)d\vartheta =$$

$$= R_0^2 \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{2\pi} \left[R_0\,\frac{1-\cos 2\varphi}{2}\sin\vartheta(1-\cos^2\vartheta) + \frac{\sin 2\vartheta}{2} - R_0\cos^2\vartheta\sin\vartheta\right]d\varphi$$

$$= R_0^2 \int_0^{\frac{\pi}{2}} \left[R_0\pi\sin\vartheta(1-\cos^2\vartheta) + \pi\sin(2\vartheta) - 2\pi R_0\cos^2\vartheta\sin\vartheta\right]d\vartheta$$

$$= R_0^3\pi\int_0^1 (1-t^2)\,dt - R_0^3 2\pi\int_0^1 t^2\,dt + R_0^2\pi\left(-\frac{1}{2}\right)\cos(2\vartheta)\Big|_0^{\frac{\pi}{2}} =$$

$$= R_0^3\pi\left(1-\frac{1}{3}\right) - \frac{2\pi R_0^3}{3} + \pi R_0^2 = \underline{\pi R_0^2}$$

$$x = R_0 \cos t \quad \Rightarrow \quad \dot{x} = -R_0 \sin t$$
$$y = R_0 \sin t \qquad \dot{y} = R_0 \cos t \qquad t \in [0, 2\pi]$$
$$z = 0 \qquad \dot{z} = 0$$

$$\underline{\int_{\partial\Sigma} \vec{R}\, d\vec{r}} = \int_{\partial\Sigma} (zy,\ x,\ x-y+z)\, d\vec{r} = \int_0^{2\pi} (0,\ R_0\cos t,\ R_0(\cos t - \sin t))(-R_0 \sin t,\ R_0\cos t,\ 0)\, dt =$$

$$= \int_0^{2\pi} R_0^2 \cos^2 t\, dt =$$

$$= R_0^2 \int_0^{2\pi} \frac{1+\cos(2t)}{2}\, dt =$$

$$\underline{= \pi R_0^2}$$

D ... polkrogla: $\partial D = \Sigma \cup \Sigma_1$ ← $K(0,R_0) \subseteq \mathbb{R}^2$

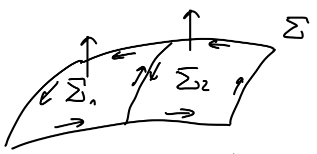
$$\iint_{\partial D} \mathrm{rot}\vec{R}\, d\vec{S} \overset{\text{Gauss}}{=} \iiint_D 0\, dV = 0$$

$$0 = \iint_{\partial D} \mathrm{rot}\vec{R}\, d\vec{S} = \iint_{\Sigma \cup \Sigma_1} \mathrm{rot}\vec{R}\, d\vec{S} = \iint_{\Sigma} \mathrm{rot}\vec{R}\, d\vec{S} + \iint_{\Sigma_1} \mathrm{rot}\vec{R}\, d\vec{S} = 0$$

$$\vec{N} = (0,0,1)$$

$$\Rightarrow \underline{\iint_{\Sigma} \mathrm{rot}\vec{R}\, d\vec{S}} = \iint_{\Sigma_1} \mathrm{rot}\vec{R} \cdot (0,0,1)\, dS = \iint_{\Sigma_1} 1\, dS = \underline{\pi R_0^2}$$

$$\overline{\Sigma} = \overline{\Sigma_1} \cup \overline{\Sigma_2}$$



Vektorsko polje je $\vec{R} \in C^1(\overline{\Sigma})$.
Naj velja Stokesov izrek za $\vec{R}$ na $(\Sigma_1, \vec{N})$ in $\vec{R}$ na $(\Sigma_2, \vec{N})$:

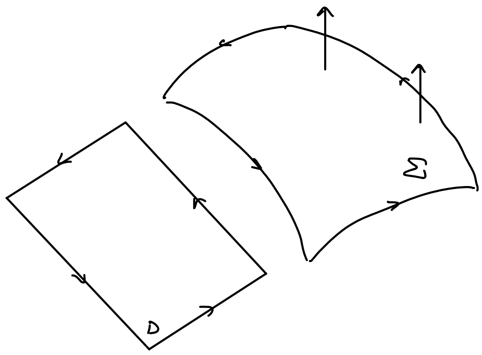
$$\int_{\partial\Sigma_1} \vec{R}\, d\vec{r} = \iint_{\Sigma_1} \mathrm{rot}\vec{R}\, d\vec{S} \quad \text{in} \quad \int_{\partial\Sigma_2} \vec{R}\, d\vec{r} = \iint_{\Sigma_1} \mathrm{rot}\vec{R}\, d\vec{S}.$$

Velja: $$\iint_{\overline{\Sigma}_1} \mathrm{rot}\vec{R}\, d\vec{S} + \iint_{\overline{\Sigma}_2} \mathrm{rot}\vec{R}\, d\vec{S} = \iint_{\overline{\Sigma} = \overline{\Sigma}_1 \cup \overline{\Sigma}_2} \mathrm{rot}\vec{R}\, d\vec{S}.$$

Integrala $\vec{R}$ po skupnem delu roba Σ_1 in Σ_2 se zaradi izbire orientacije izničita.
Torej je: $$\int_{\partial\Sigma_1} \vec{R}\, d\vec{r} + \int_{\partial\Sigma_2} \vec{R}\, d\vec{r} = \int_{\partial\Sigma} \vec{R}\, d\vec{r}.$$

Naj bo Σ graf nad xy-ravnino: $\Sigma = \{(x,y,f(x,y));\ (x,y)\in D\}$, D omejena odprta množica v $\mathbb{R}^2$, katere rob je sestavljen iz končnega števila odsekoma gladkih krivulj

orientacija Σ: $\vec{N} = \dfrac{(-f_x, -f_y, 0)}{\sqrt{1+f_x^2+f_y^2}}$



$f \in C^1(D)$ $\qquad z = f(x,y)$

$\qquad\qquad dz = f_x dx + f_y dy$

$\vec{R} = (X,Y,Z) \in C^1(\overline{\Sigma})$

$$\int_{\partial\Sigma} \vec{R}\, d\vec{r} = \int_{\partial\Sigma} X dx + Y dy + Z dz =$$

$$= \int_{\partial\Sigma} (X + Z\cdot f_x) dx + (Y + Z\cdot f_y) dy \overset{\text{Gauss}}{=}$$

$$= \iint_D \left[(Y+Zf_y)_x - (X+Zf_x)_y\right] dx dy =$$

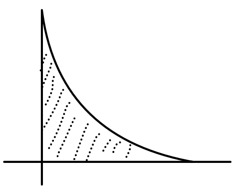
$$= \iint_D \left[\left(Y_x + Y_z f_x + Z_x f_y + Z_z f_x f_y + Z f_{xy}\right) - \left(X_y + X_z f_y + Z_y f_x + Z_z f_y f_x + Z f_{xy}\right)\right] dx dy =$$

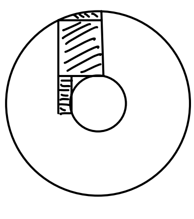
$$= \iint \left[(Y_x - X_y) + (-f_y)(X_z - Z_x) + (-f_x)(Z_y - Y_z)\right] dx\, dy =$$

$$= \iint_D \operatorname{rot}\vec{R}\cdot\vec{N} \underbrace{\sqrt{1+f_x^2+f_y^2}}_{dS}\, dx dy$$

Dokaz (Gaussov izrek)

Za odprte množice posebnih oblik.
$D \overset{odp}{\subseteq} \mathbb{R}^3$, omejena in taka, da za vsako premico vzporedno z eno od koordinatnih osi, in seka D, se na ∂D v natanko dveh točkah iz D.
Oziroma D nad vsako od koordinatnih ravnin leži med grafoma dve funkcij dveh spremenljivk.
Če je D konveksna množica, ima to lastnost.





Radi bi dokazali:
$\vec{R} \in C^1(\overline{D})$ vektorsko polje
$\vec{R} = (X, Y, Z)(x,y,z)$
$\vec{N}$ je enotska zunanja normala na ∂D: $\vec{N} = (N^x, N^y, N^z)$

Trdimo: $\displaystyle\iint_{\partial D} \vec{R}\cdot\vec{N}\, dS = \iiint_D \operatorname{div}\vec{R}\, dV.$

$$\underbrace{I}_{=} = \iint_{\partial D} (XN^x + YN^y + ZN^z)\, dS = \iiint_D (X_x + Y_y + Z_z)\, dV$$

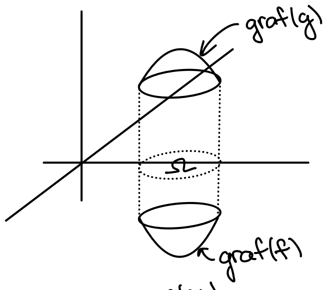
Trdimo, da velja: (1) $\iint_{\partial D} XN^x dS = \iiint_D X_x dV$ (2) $\iint_{\partial D} YN^y dS = \iiint_D Y_y dV$

(3) $\iint_{\partial D} ZN^z dS = \iiint_D Z_z dV$

Nad ravnino (x,y) leži D med dvema grafoma.

$\exists\, \Omega \in \mathbb{R}^2$. $f,g: \overline{\Omega} \to \mathbb{R}$ C^1 funkciji

$D = \{(x,y,z) \in \mathbb{R}^3;\ (x,y) \in \Omega,\ f(x,y) < z < g(x,y)\}$



$\partial D = \text{graf}(f) \cup \text{graf}(g) \cup$ "navpični del"

$$\iiint_D Z_z dV = \iint_\Omega \left(\int_{f(x,y)}^{g(x,y)} Z_z dz \right) dx dy = \iint_\Omega [Z(x,y,g(x,y)) - Z(x,y,f(x,y))] dx dy$$

$$\iint_{\partial D} Z \cdot N^z dS = \iint_{\text{graf}(g)} Z \cdot N^z dS + \iint_{\text{graf}(g)} ZN^z dS + \iint_{\text{"vmesni del"}} Z \cdot N^z dS = (*)$$

$N^z = ?$: graf(g): $\vec{N} = \frac{(-g_x, -g_y, 1)}{\sqrt{1+g_x^2+g_y^2}}$ $N^z = \frac{1}{\sqrt{1+g_x^2+g_y^2}}$

graf(f): $\vec{N} = \frac{(f_x, f_y, -1)}{\sqrt{1+f_x^2+f_y^2}}$ $N^z = \frac{-1}{\sqrt{1+f_x^2+f_y^2}}$

vmesni del: $N^z = 0$

$$(*) = \iint_\Omega Z(x,y,f(x,y)) \left(\frac{-1}{\sqrt{1+f_x^2+f_y^2}} \right) \overbrace{\sqrt{1+f_x^2+f_y^2}\, dx dy}^{dS} + \iint_\Omega Z(x,y,g(x,y)) \frac{1}{\sqrt{1+g_x^2+g_y^2}} \overbrace{\sqrt{1+g_x^2+g_y^2}\, dx dy}^{dS} = I$$

$\square$

POSLEDICA: (Greenovi identiteti)

Naj bo $D \subseteq \mathbb{R}^3$ omejena odprta množica z odsekoma gladkim robom, sestavljenim iz končnega števila odsekoma gladkih sklenjenih ploskev, orientiranih z zunanjo normalo. Naj bosta $u, v \in C^2(\overline{D})$.

Tedaj velja:

$$(1) \iint_{\partial D} u \frac{\partial v}{\partial \vec{n}} dS = \iiint_D \vec{\nabla} u \cdot \vec{\nabla} v\, dV + \iiint_D u \cdot \Delta v\, dV$$

enotska zunanja normala na ∂D

smerni odvod v v smeri zunanje normale $\vec{n}$: $\vec{\nabla} v \cdot \vec{n}$

$$(2) \iint_{\partial D} u \frac{\partial v}{\partial \vec{n}} dS - \iint_{\partial D} v \frac{\partial u}{\partial \vec{n}} dS = \iiint_D (u \Delta v - v \Delta u)\, dV$$

POSLEDICA: $v = 1$, $\Delta u = 0$ (u harmonična) $\Rightarrow \iint_{\partial D} \frac{\partial u}{\partial \vec{n}} dS = 0 \Big/ \iint_{\partial D} \vec{\nabla} u\, d\vec{S} = 0$.

Dokaz:

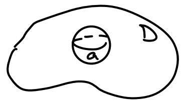
(1) $\Rightarrow$ (2) očitno

(1): $\vec{R} = u\cdot\vec{\nabla}v = (uv_x, uv_y, uv_z)$

$$\operatorname{div}\vec{R} = (uv_x)_x + (uv_y)_y + (uv_z)_z = u_xv_x + u_yv_y + u_zv_z + uv_{xx} + uv_{yy} + uv_{zz} = \vec{\nabla}u\vec{\nabla}v + u\Delta v$$

$$\iint_{\partial D} u\frac{\partial v}{\partial \vec{n}}\,dS \overset{\text{def}}{=} \iint_{\partial D} u\vec{\nabla}v\,d\vec{S} \overset{\text{Gauss}}{=} \iiint_D \operatorname{div}(u\vec{\nabla}v)\,dV = \iiint_D (\vec{\nabla}u\vec{\nabla}v + u\Delta v)\,dV \quad \square$$

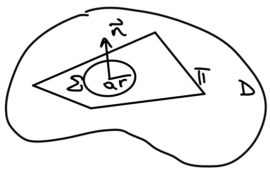
BREZKOORDINATNE DEFINICIJE:



$\operatorname{div}\vec{R}$: Naj bo $\vec{R}\in C^1(D)$, $a\in D$, $r>0$, $\overline{K(a,r)}\subseteq D^{odp}\subseteq\mathbb{R}^3$

Po Gaussovem izreku:
$$\iint_{\partial K(a,r)} \vec{R}\,d\vec{S} = \iiint_{K(a,r)} \operatorname{div}\vec{R}\,dV = \operatorname{div}\vec{R}(\xi)\cdot V(K(a,r)) \;;\quad \xi\in K(a,r)$$

Dobimo: $$(\operatorname{div}\vec{R})(a) = \lim_{r\searrow 0}\frac{1}{V(K(a,r))}\iint_{S(a,r)}\vec{R}\,d\vec{S}$$



$\operatorname{rot}\vec{R}$: $\vec{R}\in C^1(D)$, $D^{odp}\subseteq\mathbb{R}^3$, $a\in D$, $|\vec{n}|=1$

Π ravnina skozi a z normalo $\vec{n}$

Σ - krog s središčem v a, polmerom $r>0$, ki leži v Π, orientacija $\vec{n}$

Po Stokesovem izreku:
$$\int_{\partial\Sigma}\vec{R}\,d\vec{r} = \iint_\Sigma \operatorname{rot}\vec{R}\,d\vec{S} = \iint_\Sigma (\operatorname{rot}\vec{R}\cdot\vec{n})\,dS = (\operatorname{rot}\vec{R})(\xi)\cdot\vec{n}\cdot P(\Sigma)$$

Dobimo: $$(\operatorname{rot}\vec{R})(a)\cdot\vec{n} = \lim_{r\searrow 0}\frac{1}{P(\Sigma)}\int_{\partial\Sigma}\vec{R}\,d\vec{r}$$

{NI SNOV IZPITA!}

DIFERENCIALNE FORME

0-FORME: gladke funkcije (skalarna polja) (na D: vektorski prostor)

$f \in C^\infty(D)$

diferencial: $df = f_x dx + f_y dy + f_z dz$

dx ... diferencial projekcije $(x,y,z) \mapsto x$
dy ... diferencial projekcije $(x,y,z) \mapsto y$
dz ... diferencial projekcije $(x,y,z) \mapsto z$

V vsaki točki $(x,y,z) \in D$ je $f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz = df$ funkcional:

$\vec{v} = (v_1, v_2, v_3)$: $df(v_1, v_2, v_3) = f_x(x,y,z)v_1 + f_y(x,y,z)v_2 + f_z(x,y,z)v_3 = (\vec{\nabla} f)(x,y,z) \cdot \vec{v}$

POSEBEN PRIMER

1-FORME: $\omega = X(x,y,z)\,dx + Y(x,y,z)\,dy + Z(x,y,z)\,dz$
baza: dx, dy, dz (na D: vektorski prostor)

Nekatere 1-forme so diferenciali 0-form $\omega = df$.
Če je 1-forma take oblike, je eksaktna.

2-FORME: vektorski prostor: ($\wedge$ – vnanji produkt / "wedge product")
baza: $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$
pravila: $\wedge$ antikomutativen na 1-formah

$dz \wedge dy = -dy \wedge dz$ $\qquad dx \wedge dx = 0$
$dx \wedge dz = -dz \wedge dx$ $\qquad dy \wedge dy = 0$
$dy \wedge dx = -dx \wedge dy$ $\qquad dz \wedge dz = 0$

ω 1-forma in $\nu = Adx + Bdy + Cdz$:

$\rightarrow \underbrace{\omega \wedge \nu}_{\text{2-forma}} = XBdx \wedge dy + XCdx \wedge dz + AZ dz \wedge dx - AY dx \wedge dy + YC dy \wedge dz - BZ dy \wedge dz =$
$= (YC - BZ)dy \wedge dz + (AZ - XC)dz \wedge dx + (XB - AY)dx \wedge dy$

ODVOD 1-FORME: $\omega = Xdx + Ydy + Zdz$

$\rightarrow \underbrace{d\omega}_{\text{2-forma}} = dX \wedge dx + dY \wedge dy + dZ \wedge dz =$
$= (X_x dx + X_y dy + X_z dz) \wedge dx + (Y_x dx + Y_y dy + Y_z dz) \wedge dy + (Z_x dx + Y_y dy + Z_z dz) \wedge dz =$
$= (Z_y - Y_z)dy \wedge dz + (X_z - Z_x)dz \wedge dx + (Y_x - X_y)dx \wedge dy$

$d^2\omega = 0$
$\operatorname{Im} d_1 \leq \operatorname{Ker} d_2$

3-FORME: $D\, dx \wedge dy \wedge dz$; $D \in C^\infty(\Omega)$
baza: $dx \wedge dy \wedge dz$

1-forma: $\nu = Xdx + Ydy + Zdz$
2-forma: $\omega = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$

$\rightarrow \underbrace{\nu \wedge \omega}_{\text{3-forma}} = (Xdx + Ydy + Zdz) \wedge (Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy) =$
$= (XA + YB - ZC)dx \wedge dy \wedge dz$

ODVOD 2-FORME: $\omega = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$

$\rightarrow \underbrace{d\omega}_{\text{3-forma}} = d(Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy) =$
$= dA \wedge dy \wedge dz + dB \wedge dz \wedge dx + dC \wedge dx \wedge dy =$
$= (A_x + B_y + C_z)dx \wedge dy \wedge dz$

0-forme: $u \in C^\infty(\Omega) \xmapsto{d_1} du = u_x dx + u_y dy + u_z dz$

1-forme: $\nu = X dx + Y dy + Z dz \xmapsto{d_2} (Z_y - Y_z) dy \wedge dz + (X_z - Z_x) dz \wedge dx + (Y_x - X_y) dx \wedge dy$

2-forme: $\omega = A dy \wedge dz + B dz \wedge dx + C dx \wedge dy \xmapsto{d_3} (A_x + B_y + C_z) dx \wedge dy \wedge dz$

3-forme: $\lambda = D dx \wedge dy \wedge dz \xmapsto{d_4} 0$

$\rightarrow$ $d_2 \circ d_1 = 0$ $\quad Im d_1 \subseteq Ker d_2$
$d_3 \circ d_2 = 0$ $\quad Im d_2 \subseteq Ker d_3$ $\qquad d^2 = 0$
$d_4 \circ d_3 =$ $\quad Im d_3 \subseteq Ker d_4$

$0 \xrightarrow{d}$ 0-forme $\xrightarrow{d_1}$ 1-forme $\xrightarrow{d_2}$ 2-forme $\xrightarrow{d_3}$ 3-forme $\xrightarrow{d_4}$ 0 (višje forme)

trivialen vektorski prostor   trivialen vektorski prostor

Če velja: • $\omega = d\nu \Rightarrow \omega$ eksaktna forma;
• $d\omega = 0 \Rightarrow \omega$ sklenjena forma.

eksaktne 0-forme: 0
sklenjene 0-forme: $du = 0$ ← funkcije, konstantne na povezanih komponentah
$\cong \mathbb{R}^n$ (n število komponent)

STOKESOV IZREK: $\int_{\partial\Sigma} \underset{\text{1-forma}}{\nu} = \iint_{\Sigma} \underset{\text{2-forma}}{d\nu}$

GAUSSOV IZREK: $\iint_{\partial D} \underset{\text{2-forma}}{\omega} = \iiint_{D} \underset{\text{3-forma}}{d\omega}$

zaporedje vektorskih prostorov: $\cdots 0 \xrightarrow{d_0} V_0 \xrightarrow{d_1} V_1 \xrightarrow{d_2} V_2 \xrightarrow{d_3} V_3 \xrightarrow{d_4} 0 \cdots$

V_0: 0-forme, V_1: 1-forme, V_2: 2-forme, V_3: 3forme

KOVERIŽNI KOMPLEKS: $Im d_j \subseteq Ker d_{j+1}$
$d^2 = 0$ ← sklenjene forme ($d\omega = 0$)
eksaktne forme ($\omega = d\nu$)

$\Omega^{odp} \subseteq \mathbb{R}^3$ zvezdastno območje (npr. $K(0,R)$): $Im d_j = Ker d_{j+1}$ za $\forall j$

tak koverižni kompleks je eksakten

V splošnem tvorimo: $Ker d_{j+1} / Im d_j = H^j(\Omega)$
← KOHOMOLOGIJA

IZRAŽAVA OPERATORJA Δ V KRIVOČRTNIH PRAVOKOTNIH KOORDINATAH,

1) SFERIČNE: $(r,\varphi,\vartheta)\mapsto(r\cos\varphi\sin\vartheta,\ r\sin\varphi\sin\vartheta,\ r\cos\vartheta)$

2) CILINDRIČNE: $(r,\varphi,z)\mapsto(r\cdot\cos\varphi,\ r\cdot\sin\varphi,\ z)$

3) POLARNE: $(r,\varphi)\mapsto(r\cos\varphi,\ r\sin\varphi,\ 0)$

$$\vec r(\underbrace{u_1,u_2,u_3}_{\text{nove koordinate}})=(x,y,z)$$

KOORDINATNE KRIVULJE:
$$t\mapsto\vec r(t,u_2,u_3)$$
$$t\mapsto\vec r(u_1,t,u_3)$$
$$t\mapsto\vec r(u_1,u_2,t)$$

$H_1=|\vec r_{u_1}|$
$H_2=|\vec r_{u_2}|$
$H_3=|\vec r_{u_3}|$

$\rightarrow$ odvodi: $\vec r_{u_1},\vec r_{u_2},\vec r_{u_3}$

V presečiščih se sekajo vse tri krivulje paroma pravokotno:
$$\vec r_{u_1}\cdot\vec r_{u_2}=\vec r_{u_2}\cdot\vec r_{u_3}=\vec r_{u_1}\cdot\vec r_{u_3}=0$$
V vsaki točki je to ortogonalna baza (niso nujno normirani):

(1) $(\cos\varphi\sin\vartheta,\ \sin\varphi\sin\vartheta,\ \cos\vartheta)\rightarrow H_1=1$
$(-r\sin\varphi\sin\vartheta,\ r\cos\varphi\sin\vartheta,\ 0)\rightarrow H_2=r\sin\vartheta$
$(r\cos\varphi\cos\vartheta,\ r\cos\varphi\cos\vartheta,\ -r\sin\vartheta)\rightarrow H_3=r$
$\Rightarrow H_1\cdot H_2\cdot H_3=r^2\sin\vartheta$ $\leftarrow$ sferične

(2) $(\cos\varphi,\ \sin\varphi,\ 0)\rightarrow H_1=1$
$(-r\sin\varphi,\ r\cos\varphi,\ 0)\rightarrow H_2=r$
$(0,0,1)\rightarrow H_3=1$
$\Rightarrow H_1 H_2\cdot H_3=r$ $\leftarrow$ cilindrične

$$D\vec r=\left[\vec r_{u_1}\ \vec r_{u_2}\ \vec r_{u_3}\right]=\left[H_1\frac{\vec r_{u_1}}{H_1}\quad H_2\frac{\vec r_{u_2}}{H_2}\quad H_3\frac{\vec r_{u_3}}{H_3}\right]$$

$$J\vec r=H_1H_2H_3\cdot\underbrace{\det\left[\frac{\vec r_{u_1}}{H_1}\ \frac{\vec r_{u_2}}{H_2}\ \frac{\vec r_{u_3}}{H_3}\right]}_{\text{ortogonalna matrika}\ =\ \pm1}=\pm H_1H_2H_3$$

Definiramo: $\vec\eta_1=\frac{1}{H_1}\vec r_{u_1}$
$\vec\eta_2=\frac{1}{H_2}\vec r_{u_2}$
$\vec\eta_3=\frac{1}{H_3}\vec r_{u_3}$

V vsaki točki je to ortonormirana baza.

(1) $\vec\eta_1=(\cos\varphi\sin\vartheta,\ \sin\varphi\sin\vartheta,\ \cos\vartheta)$
$\vec\eta_2=(-\sin\varphi,\ \cos\varphi,\ 0)$
$\vec\eta_3=(\cos\varphi\cos\vartheta,\ \sin\varphi\cos\vartheta,\ -\sin\vartheta)$

(2) $\vec\eta_1=(\cos\varphi,\ \sin\varphi,\ 0)$
$\vec\eta_2=(-\sin\varphi,\ \cos\varphi,\ 0)$
$\vec\eta_3=(0,0,1)$

$\Delta u=u_{xx}+u_{yy}+u_{zz}$ $\leftarrow$ funkcija u v novih koordinatah

$(u(x,y,z)=x^2+y^2+z^2\rightarrow U(r,\varphi,\vartheta)=r^2)$

$$u(x,y,z)\longrightarrow U(u_1,u_2,u_3)=u(\vec r(u_1,u_2,u_3))$$

Odvajamo: $dU=du\circ d\vec r\ \rightarrow\ \frac{\partial U}{\partial u_1}=\underbrace{du\cdot\vec r_{u_1}}_{\text{funkcional na vektorju}}=\underbrace{\vec\nabla u\cdot\vec r_{u_1}}_{\text{skalarni produkt}}$

$$\rightarrow \quad \frac{1}{H_1}\frac{\partial U}{\partial u_1} = \vec{\nabla} u \cdot \vec{\eta}_1$$
$$\frac{1}{H_2}\frac{\partial U}{\partial u_2} = \vec{\nabla} u \cdot \vec{\eta}_2$$
$$\frac{1}{H_3}\cdot\frac{\partial U}{\partial u_3} = \vec{\nabla} u \cdot \vec{\eta}_3$$

$$\Rightarrow \vec{\nabla} u = \frac{1}{H_1}\frac{\partial U}{\partial u_1}\vec{\eta}_1 + \frac{1}{H_2}\frac{\partial U}{\partial u_2}\vec{\eta}_2 + \frac{1}{H_3}\frac{\partial U}{\partial u_3}\vec{\eta}_3$$

$$\Rightarrow \boxed{\vec{\nabla} = \frac{1}{H_1}\frac{\partial}{\partial u_1}\vec{\eta}_1 + \frac{1}{H_2}\frac{\partial}{\partial u_2}\vec{\eta}_2 + \frac{1}{H_3}\frac{\partial}{\partial u_3}\vec{\eta}_3}$$

Primer:

SFERIČNE
$$\left\{\begin{aligned} \vec{\nabla}(r^2) &= \tfrac{1}{1}2r(\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta) + \tfrac{1}{r\sin\vartheta}\cdot 0\cdot(-\sin\varphi, \cos\varphi, 0) + \\ &\quad + \tfrac{1}{r}\cdot 0\cdot(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, -\sin\vartheta) = \\ &= 2(r\cos\varphi\sin\vartheta, r\sin\varphi\sin\vartheta, r\cos\vartheta) \end{aligned}\right.$$

CILINDRIČNE
$$\left\{\begin{aligned} \vec{\nabla}(r^2) &= \tfrac{1}{1}\cdot 2r(\cos\varphi, \sin\varphi, 0) + \tfrac{1}{r}\cdot 0\cdot(-\sin\varphi, \cos\varphi, 0) + \tfrac{1}{1}\cdot 0\cdot(0,0,1) = \\ &= 2(r\cos\varphi, r\sin\varphi, 0) \end{aligned}\right.$$

TRDITEV: (1) Naj bo $v \in C^1(\Omega)$ in $\vec{F} \in C^1(\Omega)$. Potem je:
$$\operatorname{div}(v\vec{F}) = v \operatorname{div}\vec{F} + \vec{\nabla} v \cdot \vec{F}.$$

(2) Naj bo $\vec{A}, \vec{B} \in C^1(\Omega)$. Potem je:
$$\operatorname{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\operatorname{rot}\vec{A} - \vec{A}\operatorname{rot}\vec{B}.$$

Dokaz: DN

$$\Delta u = \operatorname{div}(\operatorname{grad} u)$$
$$\vec{R} = R_1\vec{\eta}_1 + R_2\vec{\eta}_2 + R\vec{\eta}_3$$
$$\operatorname{div}\vec{R} = \operatorname{div}(R_1\vec{\eta}_1) + \operatorname{div}(R_2\vec{\eta}_2) + \operatorname{div}(R\vec{\eta}_3) = (*)$$

Stranska ugotovitev: poseben primer: $U(u_1, u_2, u_3) = u_1$
$\vec{\nabla} u = \frac{1}{H_1}\vec{\eta}_1$ ← POTENCIALNO POLJE

$$\rightarrow \underbrace{\frac{1}{H_1}\vec{\eta}_1, \frac{1}{H_2}\vec{\eta}_2, \frac{1}{H_3}\vec{\eta}_3}_{\text{potencialna polja}} \Rightarrow \operatorname{rot}(\dots) = \vec{0}$$

Denimo, da je ortonormirana baza $\{\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3\}$ pozitivno orientirana, in
$\vec{\eta}_1\times\vec{\eta}_2 = \vec{\eta}_3$, $\vec{\eta}_2\times\vec{\eta}_3 = \vec{\eta}_1$, $\vec{\eta}_3\times\vec{\eta}_1 = \vec{\eta}_2$.
$$\Rightarrow \vec{\eta}_1 = \vec{\eta}_2\times\vec{\eta}_3$$
$$\Rightarrow \frac{1}{H_2H_3}\vec{\eta}_1 = \frac{1}{H_2}\vec{\eta}_2\times\frac{1}{H_3}\vec{\eta}_3, \quad \frac{1}{H_1H_2}\vec{\eta}_3 = \frac{1}{H_1}\vec{\eta}_1\times\frac{1}{H_2}\vec{\eta}_2, \quad \frac{1}{H_1H_3}\vec{\eta}_2 = \frac{1}{H_3}\vec{\eta}_3\times\frac{1}{H_1}\vec{\eta}_1$$

$$(*) = \operatorname{div}\left(R_1H_2H_3\frac{\vec{\eta}_2}{H_2}\times\frac{\vec{\eta}_3}{H_3}\right) + \operatorname{div}\left(R_2H_1H_3\frac{\vec{\eta}_3}{H_3}\times\frac{\vec{\eta}_1}{H_1}\right) + \operatorname{div}\left(R_3H_1H_2\frac{\vec{\eta}_1}{H_1}\times\frac{\vec{\eta}_2}{H_2}\right) \overset{(1)}{=}$$
$$= \vec{\nabla}(R_1H_2H_3)\frac{\vec{\eta}_2}{H_2}\times\frac{\vec{\eta}_3}{H_3} + \vec{\nabla}(R_2H_1H_3)\frac{\vec{\eta}_3}{H_3}\times\frac{\vec{\eta}_1}{H_1} + \vec{\nabla}(R_3H_1H_2)\frac{\vec{\eta}_1}{H_1}\times\frac{\vec{\eta}_2}{H_2} =$$
$$= \vec{\nabla}(R_1H_2H_3)\frac{1}{H_2H_3}\vec{\eta}_1 + \vec{\nabla}(R_2H_1H_3)\frac{1}{H_1H_3}\vec{\eta}_2 + \vec{\nabla}(R_3H_1H_2)\frac{1}{H_1H_2}\vec{\eta}_3$$

$$\vec{R} = \overbrace{\frac{1}{H_1}\frac{\partial U}{\partial u_1}}^{R_1}\vec{\eta}_1 + \overbrace{\frac{1}{H_2}\frac{\partial U}{\partial u_2}}^{R_2}\vec{\eta}_2 + \overbrace{\frac{1}{H_3}\frac{\partial U}{\partial u_3}}^{R_3}\vec{\eta}_3 \; (= \vec{\nabla} u)$$

$$\rightarrow \operatorname{div}(\operatorname{grad} u) = \Delta u = \frac{1}{H_1H_2H_3}\frac{\partial}{\partial u_1}\left(\frac{H_2H_3}{H_1}\frac{\partial U}{\partial u_1}\right) + \frac{1}{H_1H_2H_3}\frac{\partial}{\partial u_2}\left(\frac{H_1H_3}{H_2}\frac{\partial U}{\partial u_2}\right) + \frac{1}{H_1H_2H_3}\frac{\partial}{\partial u_3}\left(\frac{H_1H_2}{H_3}\frac{\partial U}{\partial u_3}\right)$$

$$\rightarrow \boxed{\Delta u = \frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial u_1}\left(\frac{H_2H_3}{H_1}\frac{\partial U}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{H_1H_3}{H_2}\frac{\partial U}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{H_1H_2}{H_3}\frac{\partial U}{\partial u_3}\right)\right]}$$

$$x,y,z: \quad H_1=H_2=H_3=1 \quad \rightarrow \quad \Delta u = u_{xx}+u_{yy}+u_{zz}$$

CILINDRIČNE KOORDINATE: $\quad x=r\cos\varphi,\ y=r\sin\varphi,\ z=z$

$$\vec{r}(r,\varphi,z)=(r\cos\varphi,\ r\sin\varphi,\ z)$$

$$\vec{r}_r=(\cos\varphi,\ \sin\varphi,\ 0) \rightarrow H_1=1$$
$$\vec{r}_\varphi=(-r\sin\varphi,\ r\cos\varphi,\ 0) \rightarrow H_2=r$$
$$\vec{r}_z=(0,0,1) \rightarrow H_3=1$$

$$\Delta u=\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial}{\partial\varphi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)\right]=$$
$$=\frac{1}{r}\left(r\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial^2 u}{\partial\varphi^2}+r\frac{\partial^2 u}{\partial z^2}+\frac{\partial u}{\partial r}\right)$$

POLARNE KOORDINATE:

$$\Delta u=\frac{1}{r}\left(r\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial u}{\partial r}\right)$$

RADIALNO SIMETRIČNE REŠITVE: $\quad u(r,\varphi)=f(r)$

$$\Delta u=0$$
$$\Rightarrow r\frac{\partial u}{\partial r}=C \rightarrow \frac{\partial u}{\partial r}=\frac{C}{r}$$
$$\Rightarrow u=C\log r+D$$

SFERIČNE KOORDINATE:

$$\vec{r}(r,\varphi,\vartheta)=(r\cos\varphi\sin\vartheta,\ r\sin\varphi\sin\vartheta,\ r\cos\vartheta)$$

$$\vec{r}_r(r,\varphi,\vartheta)=(\cos\varphi\sin\vartheta,\ \sin\varphi\sin\vartheta,\ \cos\vartheta) \rightarrow H_1=1$$
$$\vec{r}_\varphi(r,\varphi,\vartheta)=(-r\sin\varphi\sin\vartheta,\ r\cos\varphi\sin\vartheta,\ 0) \rightarrow H_2=r\sin\vartheta$$
$$\vec{r}_\vartheta(r,\varphi,\vartheta)=(r\cos\varphi\cos\vartheta,\ r\sin\varphi\cos\vartheta,\ -r\sin\vartheta) \rightarrow H_3=r$$

$$\Delta u=\frac{1}{r^2\sin\vartheta}\left[\frac{\partial}{\partial r}\left(r^2\sin\vartheta\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\vartheta}\frac{\partial u}{\partial\varphi}\right)+\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)\right]$$

Zgled:

$$u(x,y,z)=x^2+y^2+z^2=r^2$$
$$\Delta u=\frac{1}{r^2\sin\vartheta}\left(\frac{\partial}{\partial r}\left(r^2\sin\vartheta\, 2r\right)\right)=6$$

$$u(x,y,z)=u(r,\varphi,\vartheta)=f(r)$$
$$\Delta u=0 \Leftrightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=0$$
$$r^2\frac{\partial u}{\partial r}=C \rightarrow \frac{\partial u}{\partial r}=\frac{C}{r^2} \Rightarrow u=\frac{-C}{r}+D$$

6. KOMPLEKSNA ANALIZA

KOMPLEKSNA ŠTEVILA IN FUNKCIJE

$\mathbb{C}$: $\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$ kot množica

$(a,b)\in\mathbb{R}^2$: $(a,b)\cdot(1,0)=(a,b)$
$(c,d)\in\mathbb{R}^2$ $(a,b)\cdot(c,d)=(ac-bd,\ ad+bc)$

$\rightarrow (\mathbb{R}^2,+,\cdot)$ komutativen obseg (enota $(1,0)$)

$\mathbb{R}\subseteq\mathbb{C}$: $a\in\mathbb{R}\sim(a,0)\in\mathbb{C}$
$1\sim(1,0)$

IMAGINARNA ENOTA: $i=(0,1)$
$i^2=(0,1)(0,1)=(-1,0)=-1$

i reši enačbo: $z^2+1=0$ (druga rešitev: $-i$)

Vsako kompleksno število $z=(a,b)\in\mathbb{C}$, lahko enolično zapišemo v obliki: $z=a\cdot 1+bi$
$\Rightarrow z=a+bi;\ a,b\in\mathbb{R}$.

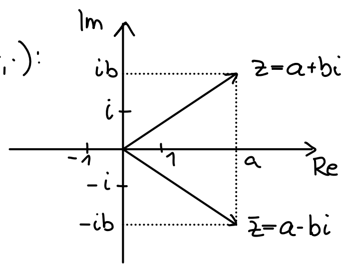
$(a+bi)\cdot(c+di)=(ac-bd)+i(ad+bc)$

$\mathbb{C}: z\longmapsto\bar{z}$ ← KONJUGIRANJE KOMPLEKSNIH ŠTEVIL
$z=(a,b)\mapsto(a,-b)=\bar{z}$
$z=a+bi\mapsto a-bi=\bar{z}$

$a=\frac{z+\bar{z}}{2}=\operatorname{Re}z$ ← realni del z
$b=\frac{z-\bar{z}}{2}=\operatorname{Im}z$ ← imaginarni del z

KOMPLEKSNA oz. GAUSSOVA RAVNINA:

$(\mathbb{C},+,\cdot)$:



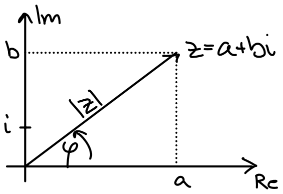
ABSOLUTNA VREDNOST:
$$|z|=\sqrt{a^2+b^2}=\sqrt{z\cdot\bar{z}}$$

NORMA $\mathbb{C}$:
1) $|z|\geq 0,\ \forall z$
2) $|z|=0\Leftrightarrow z=0$
3) $|zw|=|z|\cdot|w|,\ \forall z,w$
4) $|z+w|\leq|z|+|w|$
5) $|\operatorname{Re}z|\leq|z|;\ |\operatorname{Im}z|\leq|z|$
6) $|\bar{z}|=|z|$, metrika na $\mathbb{C}$: $d(z,w)=|z-w|$

$\Rightarrow \frac{1}{z}=\frac{\bar{z}}{z\cdot\bar{z}}\ ;\ \frac{w}{z}=\frac{w\cdot\bar{z}}{z\cdot\bar{z}}$

$\lambda\in\mathbb{R}\Rightarrow z=a+ib$
$\lambda z=\lambda a+i\cdot\lambda b$

POLARNI ZAPIS:



φ ... argument z (kot med pozitivnim delom realne osi in poltrakom $\vec{Oz}$

$\varphi \in [0, 2\pi)$

φ ni določen za 0

$a = |z|\cos\varphi$

$b = |z|\sin\varphi$

$\rightarrow z = a+bi = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$

$(\mathbb{C}, +, \cdot)$ ... KOMPLEKSNA/GAUSSOVA RAVNINA:

OBMOČJE: povezana odprta množica. $D, \Omega \subseteq \mathbb{C}$

ODPRT DISK (KROG) s središčem v točki $\alpha \in \mathbb{C}$ in polmerom $r>0$:

$\Delta(\alpha, r) = \{z \in \mathbb{C}, |z-\alpha| < r\}$

ENOTSKI DISK: $\Delta(0,1)$

1) LIMITA: $(z_n) = (z_n = a_n + ib_n) \longrightarrow \lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} a_n + i \cdot \lim\limits_{n\to\infty} b_n$

2) ZVEZNOST: $f: D \subseteq \mathbb{C} \to \mathbb{C}$

f je zvezna v α, $\alpha \in D$: $\forall \varepsilon > 0 . \exists \delta > 0 . \forall z \in \Delta(\alpha, \delta) \cap D$

$\Rightarrow f(z) \in \Delta(f(\alpha), \varepsilon)$. $(|z-\alpha| < \delta, z \in D \Rightarrow |f(z) - f(\alpha)| < \varepsilon)$

3) $f: D \subseteq \mathbb{C} \to \mathbb{C}$:

$w = f(z) = u(z) + i \cdot v(z)$ $\quad (u, v: D \to \mathbb{R})$

$z = x+iy \rightarrow u(z) = u(x,y)$

$v(z) = v(x,y)$

Velja: f je zvezna v $\alpha \in D \Leftrightarrow u, v$ zvezni v α

Zgledi:

1) $(x,y) \mapsto (x,y)$

$f(z) = z = x+iy$

$f: D \subseteq \mathbb{C} \to \mathbb{C}$ $\quad u(x,y) = x$

$(x,y) \mapsto (u(x,y), v(x,y))$ $\quad v(x,y) = y$

2) $(x,y) \mapsto (x,-y)$

$f(z) = \bar{z} = x - iy$

$u(x,y) = x$

$v(x,y) = -y$

3) $(x,y) \longmapsto (x^2+y^2, 0)$

$f(z) = |z|^2 = x^2+y^2$

$u(x,y) = x^2+y^2$

$v(x,y) = 0$

4) LIMITA FUNKCIJE: Naj bo $\alpha \in D \overset{odp}{\subseteq} \mathbb{C}$ in $f: D\setminus\{\alpha\} \to \mathbb{C}$. ← prebodena okolica

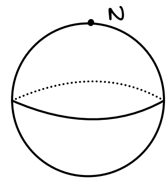
Funkcija f ima limito A, ko gre z proti α: $\lim\limits_{z\to\alpha} f(z) = A$

$\forall \varepsilon > 0 . \exists \delta > 0 . \forall z \in (\Delta(\alpha, \delta) \cap D) \setminus \{\alpha\}$, $0 < |z-\alpha| < \delta$.

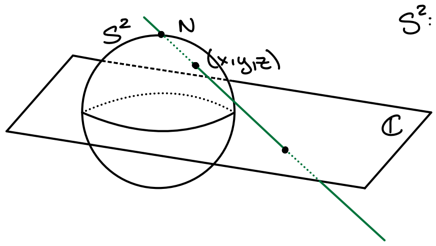
$\Rightarrow |f(z) - A| < \varepsilon$ oz. $f(z) \in \Delta(A, \varepsilon)$

5) $f(z) = u(x,y) + i \cdot v(x,y)$: velja $\lim\limits_{z\to\alpha} f(z) = A \Leftrightarrow \lim\limits_{(x,y)\to\alpha} u(x,y) + i \lim\limits_{(x,y)\to\alpha} v(x,y) = A$

6) RIEMANNOVA SFERA: $\overline{\mathbb{C}} = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$
(kompaktifikacija $\mathbb{C}$ z eno točko)



7) STEREOGRAFSKA PROJEKCIJA:



$S^2: x^2+y^2+z^2=1 \quad (\subseteq \mathbb{R}^3)$
$N(0,0,1)$

$\Phi : S^2 \setminus \{N\} \to \mathbb{C}$
$(x,y,z) \mapsto \Phi(x,y,z) = \frac{x}{1-z} + i\frac{y}{1-z}$

$\vec{s} = (x, y, z-1)$
$\vec{r}(t) = t(x,y,z-1) + (0,0,1); \quad t \in \mathbb{R}$
PREMICA SKOZI N IN (x,y,z)
Pogoj: $t(z-1)+1 = 0 \Rightarrow t = \frac{1}{1-z}$

Inverz: $\Phi^{-1}(\alpha) = \Phi^{-1}(x+iy) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$

$|\Phi^{-1}(\alpha)| = 1$

$\to \Phi$ je homeomorfizem med $S^2\setminus\{N\}$ in $\mathbb{C}$ (difeomorfizem)

$\tilde{\Phi} : S^2 \longrightarrow \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:
$\tilde{\Phi}|_{S^2\setminus\{N\}} = \Phi$
$\tilde{\Phi}(N) = \infty \qquad \to \tilde{\Phi}$ homeomorfizem

Bazične okolice točke ∞: slike s $\tilde{\Phi}$ bazičnih okolic N na S^2 so $\mathbb{C} \setminus \overline{\Delta(0,R)}$, $R>0$; bazične kolice ∞.
$(\mathbb{C}\setminus\overline{\Delta(0,R)}) \cup \{\infty\}$

HOLOMORFNE FUNKCIJE

Naj bo $D^{odp} \subseteq \mathbb{C}$ in $f : D \to \mathbb{C}$:

(1) Naj bo $\alpha \in D$. Funkcija f je v KOMPLEKSNEM SMISLU ODVEDLJIVA v α, če obstaja limita:

$$\lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} = \lim_{h\to 0} \frac{f(\alpha+h)-f(\alpha)}{h} = f'(\alpha)$$

(2) Funkcija f je HOLOMORFNA (analitična) na D, če je v kompleksnem smislu odvedljiva v vsaki točki $\alpha \in D$.

Zgledi:

1) $f(z) = \beta$
$f'(z) = 0$
$f'(\alpha) = 0 \,;\, \forall\alpha$

2) $f(z) = z$
$\alpha \in \mathbb{C}$; $f'(\alpha) = 1$, $\forall\alpha$
$\lim\limits_{h\to0}\frac{(\alpha+h)-\alpha}{h} = \lim\limits_{h\to0}\frac{h}{h} = 1$

3) $f(z) = \bar{z} \quad \alpha \in \mathbb{C}$
$\lim\limits_{h\to0}\frac{\overline{\alpha+h}-\bar{\alpha}}{h} = \lim\limits_{h\to0}\frac{\bar{h}}{h}$ ne obstaja
$(h \in \mathbb{R}\setminus\{0\} \to \frac{\bar{h}}{h} = 1$
$h \in i\mathbb{R}\setminus\{0\} \to \frac{\bar{h}}{h} = -1)$

4) $f(z) = z^2 = (x^2-y^2) + 2xyi$
$\lim\limits_{h\to0}\frac{(\alpha+h)^2-\alpha^2}{h} = \lim\limits_{h\to0}\frac{2\alpha h+h^2}{h} = 2\alpha = f'(\alpha)$

5) $f(z)=|z|^2=z\cdot\bar{z}$

$$\lim_{h\to0}\frac{|\alpha+h|^2-|\alpha|^2}{h}=\lim_{h\to0}\frac{(\alpha+h)(\bar\alpha+\bar h)-\alpha\bar\alpha}{h}=\lim_{h\to0}\frac{\bar\alpha h+\alpha\bar h+h\bar h}{h}=\lim_{h\to0}\left(\bar\alpha+\alpha\frac{\bar h}{h}+\bar h\right)=$$

$$=\bar\alpha+\lim_{h\to0}\frac{\bar h}{h}\cdot\alpha \leftarrow \text{ta limita obstaja} \Leftrightarrow \alpha=0$$

$\mathcal{O}(D)$ oz. $\mathcal{H}(D)$ ... množica holomorfnih funkcij na D.

TRDITEV: Naj bo $\alpha\in D^{odp}\subseteq\mathbb{C}$ in $f: D\to\mathbb{C}$.
Če je f v α v kompleksnem smislu odvedljiva, potem je zvezna in diferenciabilna:

$$(d_\alpha f)(h)=f'(\alpha)h$$

Dokaz:

ZVEZNOST

$$f'(\alpha)=\lim_{h\to0}\frac{f(\alpha+h)-f(\alpha)}{h} \Leftrightarrow \lim_{h\to0}\Big(\underbrace{\frac{f(\alpha+h)-f(\alpha)}{h}-f'(\alpha)}_{\beta(h)}\Big)=0 \Leftrightarrow \lim_{h\to0}\beta(h)=0$$

Dobimo: $f(\alpha+h)=f(\alpha)+f'(\alpha)h+\beta(h)\cdot h$

limita desne strani pri $h\to0$ je $f(\alpha)$, torej $\lim_{h\to0}f(\alpha+h)=f(\alpha)$

$\Rightarrow f$ je zvezna v α.

DIFERENCIABILNOST

$$(d_\alpha f)h=f'(\alpha)h\,;\quad \lim_{h\to0}\frac{|\beta(h)h|}{|h|}=0$$

$\square$

IZREK: $\mathcal{H}(D)$ je algebra nad $\mathbb{C}$.
Če sta $f,g\in\mathcal{H}(D)$, so tudi funkcije $f+g,\ fg,\ f\cdot g\in\mathcal{H}(D)$.
Če je $g(z)\neq0\ \forall z\in D$: $\frac{f}{g}\in\mathcal{H}(D)$.

Dokaz:
Pravila za odvajanje holomorfnih funkcij so formalno taka kot pravila za odvajanje na $\mathbb{R}$.

$$(f\pm g)'=f'\pm g' \qquad (f\cdot g)'=f'g+fg' \qquad \left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$$

$\square$

IZREK: Naj bo $f: D\to\Omega$ holomorfna in $g:\Omega\to\mathbb{C}$ holomorfna.
Tedaj je $g\circ f\in\mathcal{H}(D)$ in $(g\circ f)'(\alpha)=g'(f(\alpha))\,f'(\alpha)$, $\forall\alpha$.

Dokaz: (verižno pravilo)

$$D_\alpha \xrightarrow{f} \underset{\beta=f(\alpha)}{\Omega} \xrightarrow{g} \mathbb{C} \qquad \Delta(\alpha,h)\subseteq D,\ \Delta(\beta,k)\subseteq\Omega$$

Velja: $f(\alpha+h)=f(\alpha)+f'(\alpha)h+\beta_f(h)\cdot h$ in $g(\beta+k)=g(\beta)+g'(\beta)k+\beta_g(k)k$

$k=f'(\alpha)h+h\beta_f(h) \rightarrow \lim_{h\to0}k=0$

$$(g\circ f)(\alpha+h)=g(f(\alpha+h))=g(\overset{=\beta}{f(\alpha)}+k)=g(\beta)+g'(\beta)k+k\beta_g(k)=$$
$$=(g\circ f)(\alpha)+g'(f(\alpha))f'(\alpha)h+g'(f(\alpha))h\beta_f(h)+k\beta_g(k)$$

$$\lim_{h\to 0} \frac{g'(f(\alpha))h\beta_f(h)}{h} = 0$$

$$\lim_{h\to 0} \frac{k\beta_g(k)}{h} = \lim_{h\to 0} \frac{(f'(\alpha)h + h\beta_f(h))\beta_g(k)}{h} = 0$$

$\square$

OPOMBA: Ti dve izjavi se dokažeta in veljata po točkah (odvod v kompleksnem smislu).

IZREK (CAUCHY-RIEMANNOV SISTEM ENAČB):
Naj bo $f = u + v : D \to \mathbb{C}$ $(u, v : D \to \mathbb{R})$ in $\alpha = a + ib \in D$.

(1) Če je f v kompleksnem smislu odvedljiva v α, sta u in v diferenciabilni in parcialno odvedljivi v točki (a,b) in velja:
$u_x = v_y$ in $u_y = -v_x$ (CR-sistem)

(2) Če sta u, v diferenciabilni v (a,b) in zanju velja v (a,b): $u_x(a,b) = v_y(a,b)$ in $u_y(a,b) = -v_x(a,b)$ (CR-sistem), je f v α v kompleksnem smislu odvedljiva.

Dokaz:

(1): $\alpha = x_0 + y_0 i$; $h \in \mathbb{C}\setminus\{0\}$:

$$f'(\alpha) = \lim_{h\to 0} \frac{f(\alpha+h) - f(\alpha)}{h} = \lim_{h\to 0} \frac{u(\alpha+h) + iv(\alpha+h) - u(\alpha) - iv(\alpha)}{h} = \lim_{h\to 0}\left(\frac{u(\alpha+h)-u(\alpha)}{h} + i\frac{v(\alpha+h)-v(\alpha)}{h}\right) = (*)$$

(i) $h \in \mathbb{R}\setminus\{0\}$:

$$(*) = \lim_{h\to 0}\left(\frac{u(x_0+h, y_0) - u(x_0,y_0)}{h} + i\frac{v(x_0+h,y_0) - v(x_0,y_0)}{h}\right) = u_x(\alpha) + i v_x(\alpha)$$

(ii): $h = ik$, $k \in \mathbb{R}\setminus\{0\}$:

$$(*) = \lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0,y_0)}{k} + i \frac{v(x_0,y_0+k) - v(x_0,y_0)}{k} = -i u_y(\alpha) + v_y(\alpha) = f'(\alpha)$$

$\Rightarrow u_x = v_y$ in $u_y = -v_x$

(2): u, v diferenciabilni v $\alpha = x_0 + iy_0$

$$u(x_0+h, y_0+k) = u(x_0,y_0) + u_x(x_0,y_0)h + u_y(x_0,y_0)k + \delta_u(h,k)$$

$$v(x_0+h, y_0+k) = v(x_0,y_0) + v_x(x_0,y_0)h + v_y(x_0,y_0)k + \delta_v(h,k)$$

$$\to \lim_{(h,k)\to 0} \frac{f(\alpha+h+ki) - f(\alpha)}{h+ki} = \lim_{(h,k)\to 0} \frac{(u(x_0+h,y_0+k) - u(x_0,y_0)) + i(v(x_0+h,y_0+k) - v(x_0,y_0))}{h+ik} =$$

$$= \lim_{(h,k)\to 0} \frac{u_x(\alpha)h + u_y(\alpha)k + iv_x(\alpha)h + iv_y(\alpha)k + \delta(h,k)}{h+ik} =$$

$$= \lim_{(h,k)\to 0} \frac{u_x(\alpha)h + u_y(\alpha)k + i(-u_y(\alpha)h + u_x(\alpha)k)}{h+ik} = \lim_{(h,k)\to 0} \frac{u_x(\alpha)(h+ik) + iu_y(-h+ik)}{h+ik} = u_x(\alpha) - iu_y(\alpha)$$

$\square$

Zgledi:

1) $f(z) = z = x + iy$
$u(x,y) = x$
$v(x,y) = y$
$u_x = 1 = v_y$
$u_y = 0 = -v_x$

2) $f(z) = \bar{z} = x - iy$
$u(x,y) = x$
$v(x,y) = -y$
$u_x = 1 \neq -1 = v_y$

3) $f(z) = z^2 = (x^2 - y^2) + 2xyi$
$u(x,y) = x^2 - y^2$
$v(x,y) = 2xy$
$u_x = 2x = v_y$
$u_y = -2y = -v_x$

4) $f(z) = |z|^2 = x^2 + y^2$
$u(x,y) = x^2 + y^2$
$v(x,y) = 0$
$u_x = 2x$ $u_y = 2y$ $v_x = v_y = 0$

5) $f: \mathbb{C} \to \mathbb{R}$ holomorfna

$f = u + iv$; $v \equiv 0$

$$\left.\begin{array}{l} u_x = 0 \\ u_y = 0 \end{array}\right\} u = c$$

$f = 0$

Zgled: $(x,y) \mapsto (e^x \cos y, e^x \sin y)$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$$

$$u_x = e^x \cos y \qquad v_y = e^x \cos y$$
$$u_y = -e^x \sin y \qquad v_x = e^x \sin y$$

DIFERENCIALNA OPERATORJA:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$z = x + iy$:
$$\frac{\partial z}{\partial z} = \frac{1}{2}\left(1 + \frac{1}{i}i\right) = 1 \qquad \frac{\partial \bar{z}}{\partial z} = \frac{1}{2}\left(1 + \frac{1}{i}(-i)\right) = 0$$
$$\frac{\partial z}{\partial \bar{z}} = \frac{1}{2}\left(1 - \frac{1}{i}i\right) = 0 \qquad \frac{\partial \bar{z}}{\partial \bar{z}} = 1$$

$f(z) = z\bar{z}$: $\frac{\partial f}{\partial z} = \bar{z}$; $\frac{\partial f}{\partial \bar{z}} = z$

TRDITEV: Naj bo $f: D^{odp} \to \mathbb{C}$, $\alpha \in D$ in f diferenciabilna v α.
Tedaj je f v α v kompleksnem smislu diferenciabilna natanko takrat, ko velja: $\frac{\partial f}{\partial \bar{z}}(\alpha) = 0$.

Dokaz:
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) = \frac{1}{2}(u_x + iv_x + i(u_y + iv_y)) = \frac{1}{2}((u_x - v_y) + i(v_y + v_x))$$

$\uparrow$ $f = u + iv$; $u, v: D \to \mathbb{R}$

$$\frac{\partial f}{\partial \bar{z}}(\alpha) = 0 \Leftrightarrow (u_x - v_y)(\alpha) = 0 \wedge (u_y + v_x)(\alpha) = 0$$
$\Leftrightarrow$ velja CR-sistem za u in v

$\frac{\partial f}{\partial \bar{z}} = 0$ (CR-ENAČBA)

$\square$

POSLEDICA: $f: D \to \mathbb{C}$: $f \in \mathcal{H}(D) \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$ na D.

$$\begin{array}{l} \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \end{array} \Rightarrow \begin{array}{l} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \\ \frac{\partial f}{\partial y} = i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) \end{array}$$

DIFERENCIAL f: $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad z = h + ik$; $h, k \in \mathbb{R}$

$f = u + iv$; $u, v: D \to \mathbb{R}$

$$df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \to df\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \cdot \begin{bmatrix} h \\ k \end{bmatrix} = \underbrace{\begin{bmatrix} u_x \\ v_x \end{bmatrix}}_{f_x} h + \underbrace{\begin{bmatrix} v_y \\ u_y \end{bmatrix}}_{f_y} k$$

$$df\, z = f_x \frac{z + \bar{z}}{2} + f_y \frac{z - \bar{z}}{2} = (f_z + f_{\bar{z}})\frac{z + \bar{z}}{2} + (f_z - f_{\bar{z}})\frac{z - \bar{z}}{2} = f_z z + f_{\bar{z}} \bar{z}$$

$d_\alpha f = f_z(\alpha)\, d_\alpha z + f_{\bar z}(\alpha)\, d_\alpha \bar z$

f diferenciabilna: $f(\alpha+w) = f(\alpha) + d_\alpha f\, w + \delta(w) = f(\alpha) + f_z(\alpha) w + f_{\bar z}(\alpha) \bar w + \delta(w)$

$$\underbrace{\frac{f(\alpha+w)-f(\alpha)}{w}}_{\text{limita obstaja}} = f_z(\alpha) + f_{\bar z}(\alpha) \frac{\bar w}{w} + \frac{\delta(w)}{w}$$

limita obstaja $\Leftrightarrow f_{\bar z}(\alpha) = 0$

$A: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad$ $\mathbb{R}$-linearna

Kdaj bo A $\mathbb{C}$-linearna? Veljati mora $A(iv) = iAv$ $\quad$ ← množimo z i

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \to J^2 = -I \qquad AJ = JA \qquad z \mapsto iz,\ (x,y) \mapsto (-y,x)$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $a,b,c,d \in \mathbb{R}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$w = a - ib$ $\quad [a-ib]$

$(a-ib)(x+iy) = (ax+by) + i(-bx+ay)$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} ax+by \\ -bx+ay \end{bmatrix}$$

TRDITEV: Naj bo $f: D^{odp} \subseteq \mathbb{C} \to \mathbb{C}$ diferenciabilna v $\alpha \in D$. Tedaj je f v α v kompleksnem smislu odvedljiva natanko takrat, ko je $d_\alpha f$ $\mathbb{C}$-linearen.

Dokaz:

f je v α v kompleksnem smislu odvedljiva $\Leftrightarrow$ velja CR-sistem

$f = u + iv$; $u,v: D \to \mathbb{R}$ $\Leftrightarrow u_x = v_y$ in $u_y = -v_x$ v α

$d_\alpha f = \begin{bmatrix} u_x(\alpha) & u_y(\alpha) \\ v_x(\alpha) & v_y(\alpha) \end{bmatrix}$ ← $\mathbb{C}$ linearna 2×2 matrika $\Leftrightarrow \underbrace{u_x = v_y \wedge u_y = -v_x}_{\text{CR-sistem}}$ v α

□

$A(iv) = iAv$:

$$d_\alpha f = \underbrace{f_z(\alpha)\, dz}_{\mathbb{C}\text{-linearna}} + \underbrace{f_{\bar z}(\alpha)\, d\bar z}_{\mathbb{C}\text{-antilinearna}}$$

$w \mapsto f_z(\alpha) w$

ZGLEDI HOLOMORFNIH FUNKCIJ:

$f \in \mathcal{H}(\mathbb{C})$ cela holomorfna funkcija

- konstantne funkcije so holomorfne
- z holomorfna na $\mathbb{C}$

holomorfne funkcije na D tvorijo algebro $\Rightarrow$ vsi polinomi v z s kompleksnimi koeficienti so v $\mathcal{H}(\mathbb{C})$.

$p(z) = a_n z^n + \dots + a_1 z + a_0 \in \mathcal{H}(\mathbb{C})$; $a_0, \dots, a_n \in \mathbb{C}$

($z \mapsto z\bar z = x^2 + y^2$ ni holomorfna)

∘ racionalna funkcija $r(z)=\frac{p(z)}{q(z)} \in \mathcal{H}(\mathbb{C}\setminus\{x;\ q(x)=0\})$; p, q polinoma, $q \neq 0$

POTENČNE VRSTE

ŠTEVILSKE VRSTE: $(a_n)_n \in \mathbb{C}$

Vrsta $a_1+a_2+\cdots$ konvergira $\Leftrightarrow$ konvergira zaporedje delnih vsot $S_n=a_1+\cdots+a_n$ in $\lim\limits_{n\to\infty} S_n = S = \sum_{j=1}^{\infty} a_j$.

Vrsta konvergira absolutno, če konvergira vrsta $\sum_1^{\infty}|a_n|$.

Če vrsta konvergira absolutno, vrsta tudi konvergira in vrstni red členov v vrsti ni pomemben.

FUNKCIJSKE VRSTE: $(f_n)_{n\in\mathbb{N}}$; $f_n: D \to \mathbb{C}$

vrsta: $f_1+f_2+\cdots$

1) Vrsta konvergira po točkah: $\forall z\in D.\ \sum_{j=1}^{\infty} f_j(z)$ konvergira

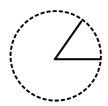
2) Vrsta konvergira enakomerno na D proti f: $\forall \varepsilon>0.\ \exists n_0.\ \forall n\geq n_0.\ \forall z\in D$
$|f(z)-S_n(z)|<\varepsilon$.

Zgled:

$f_n = z^n - z^{n-1}$ $\qquad f_1(z)=z$

$S_n = z^n$

$\lim\limits_{n\to\infty} S_n(z)=0$



Δ

3) Enakomerna konvergenca na kompaktnih podmnožicah D:

DEF: Funkcijska vrsta $\sum_1^{\infty} f_j$ konvergira enakomerno na kompaktnih podmnožicah D k funkciji f, če: $\forall K \Subset D$ kompaktno podmnožico, $\forall\varepsilon>0$, $\exists n_0$. $\forall n\geq n_0$. $\forall z\in K$: $|f(z)-S_n(z)|<\varepsilon$.

Zgled:

z^n: $D=\Delta=\Delta(0,1)$; $K \Subset D$ kompakt.
$\exists r_0\in(0,1).\ K\subseteq \overline{\Delta(0,r_0)}$ na $\overline{\Delta(0,r_0)}$: $|z^n|\leq r_0^n$

OPOMBA: $f_j \in C(D)\ \forall j \Rightarrow f\in C(D)$

OPOMBA: Velja Weierstrassov majorantni kriterij: $\sum_1^{\infty} f_j$; $f_j: D\to\mathbb{C}$
Denimo, da obstajajo $(M_j)_{j\in\mathbb{N}}$, da:
1) $|f_j(z)|<M_j$, $\forall j$, $\forall z\in D$;
2) $\sum_1^{\infty} M_j<\infty$.
Potem $\sum_1^{\infty} f_j$ konvergira enakomerno na D.

POTENČNA VRSTA:

$\alpha\in\mathbb{C}$: $(a_n)_{n=0}^{\infty}\subseteq\mathbb{C}$ $\qquad \sum_{n=0}^{\infty} a_n(z-\alpha)^n$ $\leftarrow$ potenčna vrsta s središčem v α in koeficienti $(a_n)_{n=0}^{\infty}$.

IZREK: Naj bo $\sum_0^\infty a_n(z-\alpha)^n$ potenčna vrsta. Obstaja $R\in[0,\infty]$, da velja

(1) Za vsak $0<r<R$ potenčna vrsta $\sum_0^\infty a_n(z-\alpha)^n$ konvergira absolutno in enakomerno na $\overline{\Delta(\alpha,r)}$.

(2) Za vsak $z\notin\overline{\Delta(\alpha,r)}$: $|z-\alpha|>R$, vrsta divergira.

Dokaz:

$\alpha=0$, $0<R<\infty$

(1): $|z|>R \Rightarrow \frac{1}{R}>\frac{1}{|z|}$ $\quad \limsup_{n\to\infty}\sqrt[n]{|a_n|}>\frac{1}{|z|}$

za neskončno mogo n: $\sqrt[n]{|a_n|}>\frac{1}{|z|}$; $|a_nz^n|>1$ vrsta divergira.

(2) $|z|\le r<q<R$:

$\limsup_{n\to\infty}\sqrt[n]{|a_n|}=\frac{1}{R}<\frac{1}{q}\le\frac{1}{|z|} \Rightarrow \exists n_0.\ \forall n\ge n_0: \sqrt[n]{|a_n|}<\frac{1}{q}\le\frac{1}{|z|}$

$|a_nz^n|<\left(\frac{|z|}{q}\right)^n\le\left(\frac{r}{q}\right)^n=M_n$

$\Rightarrow$ vrsta $\sum_0^\infty a_nz^n$ konvergira enakomerno na $\overline{\Delta(0,r)}$. $\square$

OPOMBA:
- R je konvergenčni polmer potenčne vrste
- $R=0$, vrsta konvergira le za $z=\alpha$.
- $R=\infty$, $\forall r>0$ vrsta konvergira enakomerno in absolutno na $\overline{\Delta(\alpha,r)}$.
- Izračun R:

$$\frac{1}{R}=\limsup_{n\to\infty}\sqrt[n]{|a_n|}$$

$R=0\Leftrightarrow \limsup\ldots=\infty$

$R=\infty\Leftrightarrow \limsup\ldots=0$

Zgledi:

(1) $\sum_0^\infty\frac{z^n}{n^n} \to a_n=\frac{1}{n^n}$ $\quad \sqrt[n]{a_n}=\frac{1}{n}\xrightarrow[n\to\infty]{}0 \to R=\infty$

(2) $\sum_0^\infty\frac{z^n}{n!} \to a_n=\frac{1}{n!}$ $\quad \limsup\frac{1}{\sqrt[n]{n!}}=0 \to R=\infty$

(3) $\sum_0^\infty n^nz^n \to R=0$

(4) $\sum_1^\infty\frac{n^n}{n} \to a_n=\frac{1}{n}$; $\lim\frac{1}{\sqrt[n]{n}}=1 \to R=1$

(5) $\sum_1^N\frac{z^n}{n}=\frac{z}{1-z}\left(\sum_1^{N-1}\frac{1-z^n}{n(n+1)}+\frac{1-z^N}{N}\right)$ konvergira za $|z|=1 \wedge z\neq1$.

Spomnimo se: $(0<R\le\infty)$: $\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)$ in $\frac{\partial}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\right)$

DEFINICIJA: Funkcija $f:D^{odp}\subseteq\mathbb{C}\to\mathbb{C}$ se da razviti v potenčno vrsto okoli točke $\alpha\in D$, če obstaja $0<r$ in potenčna vrsta $\sum_0 a_n(z-\alpha)^n$ na $\Delta(\alpha,r)$, da za vsak $z\in\Delta(\alpha,r)$ velja $f(z)=\sum_0^\infty a_n(z-\alpha)^n$.

TRDITEV: Naj bo $f(z)=\sum_0^\infty a_n(z-\alpha)^n$ potenčna vrsta na $\Delta(\alpha,r)$ $(0<r\le R)$. Tedaj je $f\in\mathcal{H}(\Delta(\alpha,r))$ in velja

$$f'(z)=\sum_1^\infty a_n\cdot n(z-\alpha)^{n-1}.$$

Dokaz:

$f(z)=\sum_0^\infty a_n(z-\alpha)^n$

$0<\rho<r$, ta vrsta enakomerno konvergira na $\overline{\Delta(\alpha,\rho)}$

$$\sum_0^\infty a_n(z-\alpha)^n=\lim_{n\to N}\underbrace{\sum_0^N a_n(z-\alpha)^n}_{P_N(z)}$$

$$\frac{\partial P_N}{\partial \bar z} = 0_N$$
$$\frac{\partial P_N}{\partial z} = \sum_1^N a_n n (z-\alpha)^{n-1} \to \sum_1^\infty n \cdot a_n (z-\alpha)^{n-1} \leftarrow$$ konvergenčni polmer te vrste je konvergenčni polmer potenčne vrste.

$$\frac{1}{R'} = \limsup \sqrt[n-1]{n|a_n|} = \frac{1}{R}$$

$\Rightarrow$ na $\overline{\Delta(\alpha,\rho)}$ konvergira $\sum_1^\infty a_n n (z-\alpha)^{n-1}$ enakomerno

$\Rightarrow f \in C^1(\Delta(\alpha,r))$ in $\frac{\partial f}{\partial \bar z} = 0 \quad \Rightarrow f \in \mathcal{H}(\Delta(\alpha,r))$

$\Rightarrow f'(z) = \frac{\partial f}{\partial z}(z) = \sum_1^\infty a_n \cdot n (z-\alpha)^{n-1} \quad \square$

POSLEDICA: Če se da $f: D \to \mathbb{C}$ v okolici vsake točke $\alpha \in D$ razviti v potenčno vrsto, je $f \in \mathcal{H}(D)$.

OPOMBA: Velja tudi obrat.

$$\begin{aligned}
f(\alpha) &= a_0 &&\Leftarrow f(z) = \sum_0^\infty a_n (z-\alpha)^n \quad ; R>0\\
f'(\alpha) &= a_1 &&\Leftarrow f'(z) = \sum_1^\infty n \cdot a_n (z-\alpha)^{n-1}\\
f''(\alpha) &= 2a_2 &&\Leftarrow f''(z) = \sum_2^\infty a_n (n-1) n (z-\alpha)^{n-2}\\
&\vdots && \qquad \vdots\\
f^{(k)}(\alpha) &= k!\, a_k &&\Leftarrow f^{(k)}(z) = \sum_k^\infty a_n \cdot n(n-1)\cdots(n-k+1)(z-\alpha)^{n-k}
\end{aligned}$$

TAYLORJEVA VRSTA ZA f NA $\Delta(\alpha,R)$: $\boxed{f(z) = \sum_0^\infty \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n}$

ELEMENTARNE FUNKCIJE V KOMPLEKSNEM

1. EKSPONENTNA FUNKCIJA:

$$e^z = \sum_0^\infty \frac{z^n}{n!} \quad ; R = \infty$$
$$(e^z)' = \sum_0^\infty \frac{z^n}{n!} = e^z$$

Za $\forall z,w \in \mathbb{C}$ velja:
$$e^{z+w} = e^z \cdot e^w \qquad \sum_n \frac{z^n}{n!} \sum_m \frac{z^m}{m!} = \sum_{n,m} \frac{z^n z^m}{n!m!} = \sum_N \Big(\sum_{n+m=N} \frac{N!}{n!m!} z^n z^m \Big) \frac{1}{N!}$$

$z,w \in \mathbb{C}$: $F(t) = e^{-t} \cdot e^{t+z+w} = F(0) = e^{z+w}$
$$F(-z) = e^z e^w$$
$$F'(t) = -e^{-t} \cdot e^{t+z+w} + e^{-t} e^{t+z+w} = 0 \quad \Rightarrow F = \text{konst.}$$

$z = x+iy$: $e^z = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$; $x,y \in \mathbb{R} \to |e^z| = e^{\operatorname{Re} z} \Rightarrow e^z \neq 0 \ \forall z$

Enačba: $e^z = 1$
$$e^x(\cos y + i\sin y) = 1 \iff e^x = 1 \iff x = 0$$
$$\cos y + i \sin y = 1 \iff \cos y = 1 \wedge \sin y = 0 \iff y = 2k\pi;\ k \in \mathbb{Z}$$
Vse rešitve: $z_k = 2k\pi i$; $k \in \mathbb{Z}$.

$w \in \mathbb{C}$, $w \neq 0$: $e^z = w$
$$|w| = |e^z| = |e^x| \iff x = \log|w|$$
$$w = |w| e^{i\arg(w)} ;\ \arg(w) \in [0, 2\pi)$$
$$\underbrace{e^x}_{} e^{iy} = |w| e^{i\arg(w)} \iff e^{i(y - \arg(w))} = 1$$
$$\iff y_k = \arg(w) + 2k\pi i;\ k \in \mathbb{Z}$$

2. SINUS, KOSINUS:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-1)^n \frac{z^{2n}}{(2n)!} + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \dots \qquad ; \quad R = \infty$$

$\cos' z = -\sin z$
$\sin' z = \cos z$

$\boxed{\cos z + i \sin z = e^{iz} \; ; \; \forall z \in \mathbb{C}}$ in $\cos z - i \sin z = e^{-iz}$

$\to \cos z = \frac{e^{iz} + e^{-iz}}{2}$

$\to \sin z = \frac{e^{iz} - e^{-iz}}{2}$

Funkciji $\cos z$ in $\sin z$ na $\mathbb{C}$ nista omejeni.

$\cosh z = \frac{e^{z} + e^{-z}}{2}$

$\sinh z = \frac{e^{z} - e^{-z}}{2}$

$\to \cosh(iz) = \cos z$ $\qquad \cos(iz) = \cosh z$
$\quad \sinh(iz) = i \sin z$ $\qquad \sin(iz) = i \sinh z$

NIČLE FUNKCIJE $\sin z$; $z \in \mathbb{C}$:

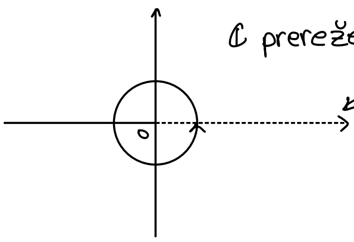
$$\sin z = \frac{e^{iz} - e^{-iz}}{2} = 0$$
$$\Leftrightarrow \quad e^{iz} - e^{-iz} = 0$$
$$\Leftrightarrow \quad e^{2iz} = 1$$
$$\Leftrightarrow \quad 2zi = 2k\pi i \quad ; \; k \in \mathbb{Z}$$
$$\boxed{z_k = k\pi}$$

Podobno določimo tudi ničle $\cos z$.

3. LOGARITEMSKA FUNKCIJA:

"inverz" eksponentni funkciji: $z \mapsto e^z$

Vsaka enačba $e^z = w$ $(w \neq 0)$ ima neskončno mnogo rešitev: $z_k = \log|w| + i\arg(w) + i2k\pi$.



$\mathbb{C}$ prerežemo vzdolž pozitivnega dela realne osi $- \mathbb{C} \setminus \{z \in [0, \infty)\}$

GLAVNA VEJA LOGARITMA

$\log w = \log|w| + i \arg w \; ; \; \arg w \in [0, 2\pi)$

(druga možnost: $(-\infty, 0)$)

4. KORENSKA FUNKCIJA:

$\sqrt{z}$

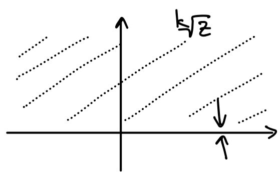
inverz kvadratni funkciji: $w \longmapsto w^2$

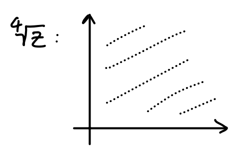
$w^2 = z = |z| e^{i\arg(z)} \quad \to \quad w = \sqrt{|z|}\, e^{i \frac{\arg z}{2}}$

GLAVNA VEJA KVADRATNEGA KORENA

$k \in \mathbb{N}$: $\sqrt[k]{z} = \sqrt[k]{|z|}\, e^{i \frac{\arg z}{k}}$

GLAVNA VEJA KVADRATNEGA KORENA





$z^{\alpha} = e^{\alpha \log z} \; ; \; \alpha \in \mathbb{C}$ $\qquad$ $(z = e^{\log z}; \; \log z = \log|z| + i\arg(z))$

$e^{\frac{1}{2}\log z} = |z|^{\frac{1}{2}} e^{i \frac{\arg z}{2}}$

Razjasnitev zapisov: $df = f_z dz + f_{\bar z} d\bar z$ ← zapis diferencialov (2x2 matrike)

$(df)h = f_z h + f_{\bar z} \bar h$ ← zapis na nivoju vektorjev (matrike delujejo na vektorjih)

Primer: $f(z) = z\bar z = x^2+y^2$ $\qquad f(z) = (u+iv)(z)$ $\quad u,v : \mathbb{C} \to \mathbb{R}$

$(x,y) \mapsto (\underbrace{x^2+y^2}_{u}, 0)$

$$df = \begin{bmatrix} 2x & 2y \\ 0 & 0 \end{bmatrix}$$

$dz: (x,y) \mapsto (x,y)$, $z \mapsto z$ $\quad dz = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\Big\}$ $d\bar z: (x,y) \mapsto (x,-y)$, $z \mapsto \bar z$ $\quad d\bar z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$a,b \in \mathbb{R}$: $z \mapsto (a+ib)z = (a+ib)(x+iy)$

$(x,y) \mapsto (ax-by) + i(bx+ay) = (ax-by, bx+ay) \to \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$f_z = \bar z = x - iy \to \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ $\qquad f_{\bar z} = z = x+iy \to \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$

$h = h_1 + h_2 i = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

$$\begin{bmatrix} 2x & 2y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} x & -y \\ y & x \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$\begin{bmatrix} 2xh_1 + 2yh_2 \\ 0 \end{bmatrix} = \begin{bmatrix} xh_1 + yh_2 \\ -yh_1 + xh_2 \end{bmatrix} + \begin{bmatrix} xh_1 + yh_2 \\ yh_1 - xh_2 \end{bmatrix}$$

KRIVULJNI INTEGRAL V $\mathbb{C}$

$\gamma \subseteq \mathbb{C}$

ODSEKOMA GLADKA ORIENTIRANA KRIVULJA

$f: \gamma \to \mathbb{C}$ zvezna funkcija

$g: \gamma \to \mathbb{C}$ zvezna

$$\int_\gamma f(z)dz \qquad \int_\gamma f(z)d\bar z \qquad \int_\gamma f(z)dz + g(z)d\bar z$$

γ gladka krivulja: (vsaj C^1)

parametrizacija: $t \mapsto z(t) = x(t) + iy(t)$; $t \in [\alpha, \beta]$ (usklajena z orientacijo)

$t \mapsto (x(t), y(t))$

DEFINICIJA:
$$\int_\gamma f(z)dz = \int_\alpha^\beta f(z(t))dz(t) = \int_\alpha^\beta f(z(t))(\dot x(t) + i\dot y(t))\,dt = \int_\alpha^\beta f(z(t))\underbrace{\dot x(t)dt}_{dx} + i\int_\alpha^\beta f(z(t))\underbrace{\dot y(t)dt}_{dy} =$$

$$= \int_\gamma f(z)dx + if(z)dy$$

$f = u + iv$; $u,v : \gamma \to \mathbb{R}$

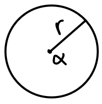
$$= \int_\gamma (u+iv)dx + i(u+iv)dy = \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy)$$

Ta definicija je neodvisna od parametrizacije, ki ohranja orientacijo.

$f: \gamma \to \mathbb{C}$ zvezna funkcija

$$\int_\gamma f(z)d\bar z = \int_\alpha^\beta (udx + vdy) + i\int_\alpha^\beta (vdx + udy)$$

Zgled:

$r>0$: 

$$\int_{|z-\alpha|=r} \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{rie^{i\varphi}}{re^{i\varphi}}\, d\varphi = 2\pi i \qquad\qquad \frac{1}{2\pi i}\int_{\partial\Delta(\alpha,r)} \frac{dz}{z-\alpha} = 1$$

$z=\alpha+re^{i\varphi}$, $\varphi\in[0,2\pi)$

$dz = rie^{i\varphi}d\varphi$

$$\int_{|z-\alpha|=r} \frac{d\bar z}{z-\alpha} = \int_0^{2\pi} \frac{-ire^{-i\varphi}}{re^{i\varphi}}\, d\varphi = -i\int_0^{2\pi} e^{-2i\varphi} d\varphi = 0 \quad -\frac{1}{2i}\, e^{-2i\varphi}\Big|_0^{2\pi}$$

OPOMBA: γ je sklenjena krivulja, če je njena končna točka enaka začetni točki.

TRDITEV: Naj bo γ odsekoma gladka krivulja, orientirana, dolžine $\ell(\gamma)$. Naj bo $f:\gamma\to\mathbb{C}$ zvezna funkcija in $\sup_\gamma|f|$ supremum $|f|$ po γ.
Tedaj je:
$$\left|\int_\gamma f(z)dz\right| \le \ell(\gamma)\cdot \sup_\gamma|f|.$$

Dokaz:
$z:[\alpha,\beta]\to\gamma$ parametrizacija
$$\left|\int_\gamma f(z)dz\right| = \left|\int_\alpha^\beta f(z(t))\,\dot z(t)dt\right| \le \int_\alpha^\beta |f(z(t))|\cdot|\dot z(t)|\,dt \le \sup_\gamma|f| \int_\alpha^\beta |\dot z(t)|dt = \sup_\gamma|f|\cdot\ell(\gamma) \quad \square$$

TRDITEV: Naj bo $D^{odp}\subseteq\mathbb{C}$ in $F\in\mathcal{H}(D)$ ter $F'\in C(D)$. Naj bo $\gamma\subseteq D$ orientirana odsekoma gladka krivulja z začetno točko $\alpha\in D$ in končno točko $\beta\in D$.
Tedaj:
$$\int_\gamma F'(z)dz = F(\beta)-F(\alpha).$$

Dokaz:
$F\in\mathcal{H}(D)$; $dF = F'(z)dz$ $\qquad F=u+iv$; $u,v: D\to\mathbb{R}$
$$\int_\gamma F'(z)dz = \int_\gamma dF = \int_\gamma du + i\,dv \overset{\int \vec\nabla u,\vec\nabla v \text{ po } \gamma}{=} (u+iv)(\beta)-(u+iv)(\alpha) = F(\beta)-F(\alpha) \quad \square$$

OPOMBA: Integral $\int_\gamma F'(z)dz$ od α do β je neodvisen od orientirane odsekoma gladke krivulje od α do β.

POSLEDICA: Če je γ sklenjena, je $\int_\gamma F'(z)dz=0$.

POSLEDICA: $n\in\mathbb{N}\cup\{0\}$, γ sklenjena v $\mathbb{C}$: $\int_\gamma z^n dz = 0$

Dokaz:
$$z^n = \left(\frac{1}{n+1}z^{n+1}\right)' \quad \square$$

POSLEDICA: γ sklenjena, $p(z)$ polinom : $\int_\gamma p(z)dz = 0$

POSLEDICA: $-n \in \mathbb{N}\cup\{1\} = \{1,2,3,\dots\}$, γ sklenjena v $\mathbb{C}\setminus\{0\}$: $\int_\gamma z^n dz = 0$

OPOMBA:
$r>0$: $\int_{\partial\Delta(0,r)} \frac{dz}{z} = 2\pi$; $\frac{1}{z} \in \mathcal{H}(\mathbb{C}\setminus\{0\})$ — "Integral" bi bil $\log z$, ki ni definiran na $\mathbb{C}\setminus\{0\}$.

GREENOVA FORMULA V KOMPLEKSNEM:

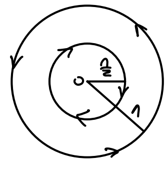
Naj bo $D^{odp} \subseteq \mathbb{C}$ omejena odprta množica z odsekoma gladkim robom, sestavljenim iz končnega števila odsekoma gladkih sklenjenih krivulj, orientiranih pozitivno glede na D.
Naj bosta $f,g: D \to \mathbb{C}$ C^1-funkciji.
Tedaj:
$$\int_{\partial D} f(z)dz + g(z)d\bar{z} = 2i \iint_D \left(f_{\bar{z}}(z) + g_z(z)\right) dx\,dy$$

Dokaz:
$$\int_{\partial D} f dz + g d\bar{z} = \int_{\partial D} f(dx+idy) + g(dx-idy) = \int_{\partial D} (f+g)dx + i(f-g)dy =$$
$$\overset{\text{Greenova formula}}{=} \iint_D \left[(if-ig)_x - (f+g)_y\right] dx\,dy = \iint_D \left[(if_x - f_y) - (ig_x + g_y)\right] dx\,dy =$$
$$= 2i \iint_D \Big(\underbrace{\left[\tfrac{1}{2}(f_x + if_y)\right]}_{f_{\bar{z}}} - \underbrace{\left[\tfrac{1}{2}(g_x + ig_y)\right]}_{g_z}\Big) dx\,dy$$
$\square$

POSLEDICA (Cauchyjev izrek):
$f \in \mathcal{H}(D)$:
$$\int_{\partial D} f(z)dz = 0$$

Dokaz:
$g=0$, $f_{\bar{z}}=0$ $\square$

Zgled:
$D = \Delta(0,1) \setminus \overline{\Delta(0,\frac{1}{2})}$

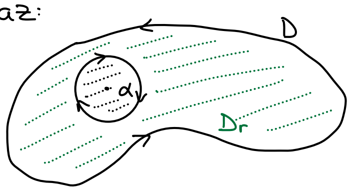


$f(z) = \frac{1}{z}$
$$\int_{\partial D} \frac{dz}{z} = \int_{|z|=1} \frac{dz}{z} + \int_{|z|=\frac{1}{2}} \frac{dz}{z} = 2\pi - 2\pi = 0$$

IZREK (Cauchyjeva formula):

Naj bo $D \subseteq \mathbb{C}$ omejena odprta množica z odsekoma gladkim robom, sestavljenim iz končnega števila odsekoma gladkih krivulj, orientiranih pozitivno glede na D. Naj bo $f \in \mathcal{H}(D) \cap C^1(\bar{D})$ in $\alpha \in D$.
Tedaj:
$$f(\alpha) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-\alpha} dz$$

Dokaz:



$\overline{\Delta(\alpha, r_0)} \subseteq D$; $r_0 > 0$; $0 < r < r_0$

$D_r = D \setminus \overline{\Delta(\alpha, r)}$

$z \mapsto \frac{f(z)}{z-\alpha} \in \mathcal{H}(D_r) \cap C^1(\overline{D}_r)$ $\qquad$ $\partial D_r = \partial D \cup \partial\Delta(\alpha, r)$

Cauchyjev izrek:
$$0 = \int_{\partial D_r} \frac{f(z)}{z-\alpha}\,dz = \int_{\partial D} \frac{f(z)}{z-\alpha}\,dz + \int_{\partial\Delta(\alpha,r)} \frac{f(z)}{z-\alpha}\,dz$$

$$\Rightarrow \int_{\partial D} \frac{f(z)}{z-\alpha}\,dz = \int_{\partial\Delta(\alpha,r)} \frac{f(z)}{z-\alpha}\,dz = \int_0^{2\pi} f(\alpha + re^{i\varphi})\,\frac{1}{re^{i\varphi}}\,re^{i\varphi}\cdot i\,d\varphi =$$

$$= i\int_0^{2\pi} f(\alpha + re^{i\varphi})\,d\varphi \xrightarrow{r \to 0} i\cdot f(\alpha)\cdot 2\pi$$

$$= \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-\alpha}\,dz = f(\alpha)$$

$\square$

OPOMBE: (1) Holomorfna funkcija je popolnoma določena z vrednostmi na ∂D.

(2) $\alpha \notin \overline{D}$: $\int_{\partial D} \underbrace{\frac{f(z)}{z-\alpha}}_{\in \mathcal{H}(D)}\,dz = 0$

(3) $\frac{1}{2\pi i}\int_{\partial\Delta(0,r)} \frac{dz}{z} = 1$; $r > 0$: $f(z) = 1$
$D = \Delta(0, r)$, $\alpha = 0$

(4) $D = \Delta(0,1) \setminus \overline{\Delta(0, \frac{1}{2})}$; $\alpha = \frac{3}{4}$, $f(z) = \frac{1}{z}$

$$\int_{\partial D} \frac{dz}{z(z-\frac{3}{4})} = 2\pi i\, f\left(\tfrac{3}{4}\right) = 2\pi i\cdot\frac{4}{3} = \frac{8}{3}\pi i.$$

(5) Če je $\varphi \in C^1(\partial D)$, ali obstaja $f \in C^1(\overline{D}) \cap \mathcal{H}(D)$, $f|_{\partial D} = \varphi$? - NE

Zgled:
$D = \Delta$; $\partial D = \partial\Delta$
$\varphi(z) = \overline{z}$
Ali obstaja $f \in \mathcal{H}(D) \cap C^1(\overline{D})$? $\rightarrow$ Da: $f(z) = \overline{z}$ na $\partial\Delta$

POSLEDICA: (lastnost povprečne vrednosti):

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\varphi})\,d\varphi \leftarrow \text{POVPREČJE } f \text{ PO } \partial\Delta(\alpha, r)$$

OPOMBA $f \in \mathcal{H}(D) \cap C^1(\overline{D})$. Velja: $f \in \mathcal{H}(D) \Rightarrow f \in C^\infty(D)$

$\frac{1}{z-1} \in \mathcal{H}(\Delta) \Rightarrow \frac{1}{z-1} \in C^\infty(\Delta)$

Pokazali bomo: $f \in \mathcal{H}(D) \Rightarrow f \in C^\infty(D)$: (i): $f \in \mathcal{H}(D) \cap C^1(D) \Rightarrow f \in C^\infty(D)$

(ii): $f \in \mathcal{H}(D) \Rightarrow \exists F \in \mathcal{H}(D) \ni: F' = f \in C(D)$
$\Rightarrow F \in C^\infty(D)$
$\Rightarrow f \in C^\infty(D)$

TRDITEV: Naj bo $D^{odp} \subseteq \mathbb{C}$; $f \in \mathcal{H}(D) \cap C^1(\overline{D})$. (kot v Greenovi formuli)
Tedaj je $f \in C^\infty(D)$ in vsi odvodi $f^{(n)} \in \mathcal{H}(D)$; $\forall n \in \mathbb{N}$, ter za vsak $\alpha \in D$ velja:

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-\alpha)^{n+1}} dz .$$

Dokaz:

Vemo:
$$f(\alpha) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-\alpha} dz \ ; \ \forall \alpha \in D.$$

$$(f(w) =) \ F(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-w} dz \ ; \ w \in D$$
← integral s parametrom

$$\frac{\partial F}{\partial w}(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^2} dz$$

$$\frac{\partial F}{\partial \overline{w}} = 0$$

$$\Rightarrow F \in C^1(D) \cap \mathcal{H}(D)$$

$$F'(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^2} dz$$

$$\frac{\partial F'}{\partial \overline{w}} = 0 \ \text{ in } \ \frac{\partial F'}{\partial w}(w) = \frac{2}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^3} dz = F''(w)$$

$\Rightarrow$ Z indukcijo: $\forall n$ obstaja: $f^{(n)}(w) = F^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^{n+1}} dz$ $\square$

POSLEDICA: Naj bo $D^{odp} \subseteq \mathbb{C}$; $f \in \mathcal{H}(D) \cap C^1(D)$.
Tedaj je $f \in C^\infty(D)$ in vsi odvodi f so holomorfne funkcije na D.

Dokaz:

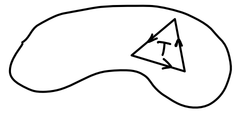
Lastnosti, kot so: $f \in C^\infty(D)$, $f \in \mathcal{H}(D)$, ..., so lokalne lastnosti.

$\alpha \in D$; $\overline{\Delta(\alpha, r_0)} \subseteq D$

Po izreku: $\forall w \in \Delta(\alpha, r_0)$: $f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial \Delta(\alpha, r_0)} \frac{f(z)}{(z-w)^{n+1}} dz$

$$\Rightarrow f \in C^\infty(\Delta(\alpha, r_0)) \rightarrow f^{(n)} \in \mathcal{H}(\Delta(\alpha, r_0))$$ $\square$

TRDITEV: (pomožna) Naj bo $D^{odp} \subseteq \mathbb{C}$ zvezdasto in $f \in C(D)$.
Denimo, da za vsak trikotnik $T \subseteq D$ velja: $\int_{\partial T} f(z) dz = 0$.

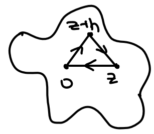


Tedaj obstaja $F \in \mathcal{H}(D)$, da je $F' = f$ in $f, F \in C^\infty(D)$.

(Iz $F \in \mathcal{H}(D)$ in $F' = f$ sledi $F \in \mathcal{H}(D) \cap C^1(D)$. $\Rightarrow f, F \in C^\infty(D)$)

Dokaz:

Naj bo D zvezdasto območje glede na 0.



$z+h \in D$

$$F(z) = \int_{[0,z]} f(\xi)\, d\xi \qquad (F \text{ dobro definiran})$$

parametrizacija: $t \mapsto z + th$; $t \in [0,1]$

Ali velja $F'(z) = f(z)$?

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\int_{[0,z+h]} f(\xi)\,d\xi - \int_{[0,z]} f(\xi)\,d\xi\right] \overset{(*)}{=} \lim_{h\to 0} \frac{1}{h}\int_{[z,z+h]} f(\xi)\,d\xi =$$

$$= \lim_{h\to 0} \frac{1}{h}\int_0^1 f(z+th)\,h\,dt = f(z)$$

f zvezna

$$(*) \quad 0 = \int_{[0,z+h]} f(\xi)\,d\xi + \int_{[z+h,z]} f(\xi)\,d\xi + \int_{[z,0]} f(\xi)\,d\xi$$

$\Rightarrow F \in \mathcal{H}(D)$, $F' = f$

$F \in C^1(D)$ in $f \in \mathcal{H}(D)$

$\Rightarrow F \in C^\infty(D) \Rightarrow f \in C^\infty(D)$ $\square$

IZREK (Morerov izrek):

Naj bo $D^{odp} \subseteq \mathbb{C}$ in $f \in C(D)$.

Denimo, da za vsak trikotnik $T \subseteq D$ velja: $\int_{\partial T} f(z)\,dz = 0$.

Tedaj je: $f \in \mathcal{H}(D) \cap C^\infty(D)$.

Dokaz:

lastnosti: $f \in C^\infty(D)$, $f \in \mathcal{H}(D)$ ... so lokalne lastnosti.

Dovolj je dokazati, da za $\forall \Delta(\alpha, r_0) \subseteq D$ velja $f \in \mathcal{H}(\Delta(\alpha,r_0)) \cap C^\infty(\Delta(\alpha,r_0))$. $\square$

Zgled:

$f(z) = \frac{1}{z}$; $D = \mathbb{C}\setminus\{0\}$

"$F(z) = \log z$" ← ni dobro definirano na $\mathbb{C}\setminus\{0\}$

IZREK (Goursatov izrek):

Naj bo $f \in \mathcal{H}(D)$.

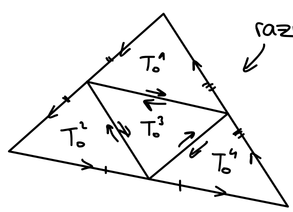
Tedaj je $f \in C^\infty(D)$.

$\big(f \in \mathcal{H}(D) \Rightarrow f \in C(D)$
$(\Rightarrow \forall T \subseteq D \ \int_{\partial T} f(\xi)\,d\xi = 0) \Rightarrow f \in C^\infty(D)\big)$

Dokaz:

Naj bo $T \subseteq D$ trikotnik v D.

$T = T_0$



razpolovišča stranic T_0

$$\text{Naj bo } I_0 = \int_{\partial T_0} f(\xi)\,d\xi = \int_{\partial T_0^1} \ldots + \int_{\partial T_0^2} \ldots + \int_{\partial T_0^3} \ldots + \int_{\partial T_0^4} \ldots$$

Vsaj en od teh integralov je $\geq \frac{|I_0|}{4}$.

Tisti trikotnik je T_1: $\left|\int_{\partial T_1} f(\xi)\,d\xi\right| \geq \frac{|I_0|}{4}$

Postopek ponovimo na T_1.

$\Rightarrow \exists T_2$: $\left|\int_{\partial T_2} f(\xi)\,d\xi\right| \geq \frac{|I_0|}{4^2}$

Dobimo zaporedje trikotnikov $\{T_n\}$: $\left|\int_{\partial T_n} f(\xi)\,d\xi\right| \geq \frac{|I_0|}{4^n}$

Premer $T_n = \frac{1}{2^n}$ (premer T_0).

$\Rightarrow \bigcap_{n=0}^{\infty} T_n = \{\alpha\} \subseteq D$

V α je f v kompleksnem smislu odvedljiva: $f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + (z-\alpha)\eta(z-\alpha)$
$\lim_{z\to\alpha} \eta(z-\alpha) = 0$

Naj bo $\varepsilon > 0$.
Ker je $\lim_{z\to\alpha} \eta(z-\alpha) = 0$, obstaja $\delta > 0$, da je $|z-\alpha| < \delta$.

$\Rightarrow |\eta(z-\alpha)| < \varepsilon$. $\exists n_0.\ \forall n \geq n_0 : \alpha \in T_n \subseteq \Delta(\alpha, \delta)$

Naj bo $n \geq n_0$: $\int_{\partial T_n} f(\xi)\,d\xi = \int_{\partial T_n} [\overbrace{\underbrace{f(\alpha)}_{0} + \underbrace{f'(\alpha)(\xi-\alpha)}_{0}}^{\text{holomorfno}} + (\xi-\alpha)\eta(\xi-\alpha)]\,d\xi$

$\left|\int_{\partial T_n} f(\xi)\,d\xi\right| \leq \int_{\partial T_n} |\xi-\alpha||\eta(\xi-\alpha)||d\xi| \leq \frac{1}{2^n}$ premer $(T_0) \cdot \varepsilon \frac{1}{2^n}$ obseg (T_0)
$\leq \frac{1}{4^n}$ obseg $(T_0)^2$

$\Rightarrow |I_0| \leq 0$
$\leq \varepsilon$ obseg $(T_0)^2$
$\Rightarrow I_0 = 0$ $\square$

$f \in \mathcal{H}(D)$ ($\Rightarrow f \in C^\infty(D)$; $f^{(n)}, \frac{\partial f}{\partial z}$ ... holomorfne)
$f = u + iv$; $u, v : D \to \mathbb{R}$, $u, v \in C^\infty(D)$
CR-sistem: $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \Rightarrow \begin{array}{l} \Delta u = 0 \\ \Delta v = 0 \end{array}$

POSLEDICA: u, v sta holomorfni funkciji na D: $\Delta u = 0 \wedge \Delta v = 0$ nad D.

Zgled:
$u(x,y) = x^2 + y^2 \to \Delta u = 4$
$\left.\begin{array}{l} \operatorname{Re} f = u \\ f \in \mathcal{H} \end{array}\right\}$ tak f ne obstaja

$u : D^{odp} \to \mathbb{R}$ holomorfna. Ali obstaja $f \in \mathcal{H}(D)$, da je $\operatorname{Re} f = u$?
(Če je u harmonična na D, potem $u \in C^\infty(D)$.)
$u_{xx} + u_{yy} = 0$ $(y'' = 0)$

Terminologija: $f = u + iv$; $f \in \mathcal{H}(D)$; $u, v : D \to \mathbb{R}$ (harmonični konjugiranki)

Ali za $u : D \to \mathbb{R}$ (harmonična) obstaja pripadajoča harmonična konjugiranka?
$\exists v : D \to \mathbb{R}$ harmonična $\ni: f = u + iv \in \mathcal{H}(D)$.

Zgled:
(1) $f(z) = z$: $u(x,y) = x$
$v(x,y) = y$

(2) $f(z) = z^2$: $u(x,y) = x^2 - y^2$
$v(x,y) = 2xy$

TRDITEV: Naj bo D zvezdasto območje v $\mathbb{C}$ in $u: D \to \mathbb{R}$ harmonična funkcija. Tedaj na D obstaja harmonična konjugiranka k u, oz. obstaja $f \in \mathcal{H}(D)$, da je $\mathrm{Re} f = u$.

Dodatek: Harmonična konjugiranka je določena do konstante natančno.

Dokaz:
Če harmonična konjugiranka v obstaja, je: $v_y = u_x$, $v_x = u_y$

$(-u_y, u_x) \leftarrow$ to mora biti $\vec{\nabla} v$

$\mathrm{rot}(-u_y, u_x, 0) = (0, 0, u_{xx} + u_{yy}) = \vec{0}$

To vektorsko polje ima potencial v:

$$v_x = -u_y$$
$$v_y = u_x \quad \square$$

Zgled:
$\mathbb{C} \setminus \{0\}$: $u(x,y) = u(z) + \log|z| = \frac{1}{2}\log(x^2+y^2)$

$u_x = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$ $\quad u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad \to \Delta u = 0$

OPOMBA: Če je D zvezdasto glede na 0: $v(x,y) = \int_{[0,(x,y)]} -u_y\,dx + u_x\,dy$

Vemo, da je vsaka funkcija $f: D \to \mathbb{C}$, $D \overset{odp}{\subseteq} \mathbb{C}$, ki se da v okolici vsake točke $\alpha \in D$ razviti v potenčno vrsto, holomorfna na D.

$$\Delta(\alpha, r);\quad f(z) = \sum_0^\infty a_n (z-\alpha)^n = \sum_0^\infty \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$$

TRDITEV: Naj bo $f \in \mathcal{H}(D)$; $\alpha \in D$ in $\overline{\Delta(\alpha,r)} \subseteq D$.
Tedaj se da f na $\Delta(\alpha,r)$ razviti v potenčno vrsto: $f(z) = \sum_0^\infty a_n (z-\alpha)^n$;
kjer je: $\frac{f^{(n)}(\alpha)}{n!} \cdot a_n = \frac{1}{2\pi i}\int_{\partial\Delta(\alpha,r)} \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$

Dokaz:
$\overline{\Delta(\alpha,r)} \subseteq D$.
Za $\forall z \in \Delta(\alpha,r)$ je $f(z) = \frac{1}{2\pi i}\int_{\partial\Delta(\alpha,r)} \frac{f(\xi)}{\xi - z}\,d\xi$

$$\frac{1}{\xi - z} = \frac{1}{(\xi-\alpha)-(z-\alpha)} = \frac{1}{(\xi-\alpha)\left(1 - \frac{z-\alpha}{\xi-\alpha}\right)}$$

$|z-\alpha| \le \rho < r \to \frac{|z-\alpha|}{|\xi-\alpha|} \le \frac{\rho}{r} < 1 \quad \forall z \in \overline{\Delta(\alpha,\rho)};\ \forall \xi \in \partial\Delta(\alpha,r)$

$\Rightarrow \frac{1}{\xi - z} = \frac{1}{\xi-\alpha}\sum_0^\infty \frac{(z-\alpha)^n}{(\xi-\alpha)^n} \leftarrow$ ta vrsta konvergira enakomerno za $z \in \overline{\Delta(\alpha,\rho)}$, $\xi \in \partial\Delta(\alpha,r)$

$\Rightarrow f(z) = \frac{1}{2\pi i}\int_{\partial\Delta(\alpha,r)} \frac{f(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\int_{\partial\Delta(\alpha,r)} \frac{f(\xi)}{\xi-\alpha}\sum_0^\infty \frac{(z-\alpha)^n}{(\xi-\alpha)^n}\,d\xi = \sum_0^\infty \left[\frac{1}{2\pi i}\int_{\partial\Delta(\alpha,r)} \frac{f(\xi)}{(\xi-\alpha)^{n+1}}\right](z-\alpha)^n = \sum_0^\infty a_n |z-\alpha|^n \quad \square$

POSLEDICA: Konvergenčni polmer $\sum_0^\infty \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$ je vsaj $d(\alpha, \partial D)$.

OSNOVNI IZREK ALGEBRE:

Naj bo p nekonstanten polinom, s kompleksnimi koeficienti: $p(z)=\sum_{j=0}^{n} a_j z^j$
$(a_n \neq 0,\ n=\deg p \geq 1;\ a_j \in \mathbb{C},\ j=0,\dots,n)$.
Tedaj ima p ničlo v $\mathbb{C}$: $\exists z_0 \in \mathbb{C} \ni: p(z_0)=0$.

Dokaz:
Naj bo p nekonstanten polinom $p(z)=a_n z^n+\dots+a_1 z+a_0$. $(n\geq 1,\ a_n\neq 0)$
Denimo, da p nima ničle na $\mathbb{C}$.
$\frac{1}{p} \in \mathcal{H}(\mathbb{C})$.
Pokazali bomo, da je $\frac{1}{p}$ omejena cela holomorfna funkcija.
Po Liouvilleovem izreku je $\frac{1}{p}$ konstantna funkcija $\Rightarrow$ p konstanten $\times$

ocena: $z\neq 0$.

$$|p(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right|$$
$$\geq |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \dots - \frac{|a_0|}{|z|^n} \right)$$

$\exists R>0$, da za $|z|>R$ velja: $|a_n| - \frac{|a_{n-1}|}{|z|} - \dots - \frac{|a_0|}{|z|^n} \geq \frac{|a_n|}{2}$

$$\rightarrow |p(z)| \geq |z|^n \frac{|a_n|}{2}$$

$$\frac{2}{|a_n| R^n} \geq \frac{2}{|a_n|}\cdot\frac{1}{|z|^n} \geq \frac{1}{|p(z)|}$$

na $\overline{\Delta(0,R)}$ je $\frac{1}{|p(z)|}$ zvezna in omejena: $\exists M \ni: \frac{1}{|p(z)|} \leq M$, $\forall |z|\leq R$

$\Rightarrow \frac{1}{|p(z)|} \leq \max\left\{M, \frac{2}{|a_n|R^n}\right\}$ in je omejena na $\mathbb{C}$ $(n\geq 1)$ $\square$

POSLEDICA: Polinom p ima toliko ničel, štetih s kratnostjo, kolikor je stopnja p:
$p(z)=a_n(z-z_1)^{k_1}\cdots(z-z_\ell)^{k_\ell}$; $k_1+\dots+k_\ell=n$.

Dokaz:
z_0 ničla: $p(z)=(z-z_0)q(z)$; $\deg q = n-1 = \deg p - 1$.
nadaljujemo z indukcijo na stopnjo polinoma ... $\square$

TRDITEV (Cauchyjeve ocene):
(pomožna) Naj bo $f\in\mathcal{H}(\Delta(0,R))$, $R>0$.
Tedaj za vsak r, $0<r<R$ in $\forall n\in\mathbb{N}\cup\{0\}$ velja ocena: $|f^{(n)}(0)| \leq \frac{n!}{r^n}\sup_{|z|=r}|f(z)|$.

Dokaz:
$$f^{(n)}(0) = \frac{n!}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{n+1}}dz$$

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi}\int_{|z|=r}\frac{|f|}{|z|^{n+1}}|dz| \leq \frac{n!}{2\pi}\cdot 2\pi r\,\frac{1}{r^{n+1}}\sup_{|z|=r}|f|$$
$\square$

OPOMBA: Velja $\sup_{\Delta(0,r)}|f| = \sup_{\partial\Delta(0,r)}|f|$.

IZREK (Liuvilleov izrek):
Naj bo $f\in\mathcal{H}(\mathbb{C})$. Denimo, da obstajata $C>0$ in $N\in\mathbb{N}\cup\{0\}$, da za vsak $z\in\mathbb{C}$ velja: $|f(z)|\leq C(1+|z|^N)$.
Tedaj je f polinom stopnje največ N.

Dokaz:

$f \in \mathcal{H}(\mathbb{C}) \Rightarrow f(z)=\sum_0^\infty a_n z^n$; $R=\infty$, $r>0$

$$|a_n| = \frac{|f^{(n)}(0)|}{n!} \le \frac{1}{r^n}\sup_{|z|=r}|f| \le C\frac{(1+r^N)}{r^n}.$$

$n\ge N$: $\lim_{r\to\infty}\frac{1+r^N}{r^n}=0 \Rightarrow a_n=0$; $\forall n>N$

$\Rightarrow f(z)=\sum_0^N a_n z^n$ ← polinom stopnje največ N □

POSLEDICA: Vsaka omejena cela holomorfna funkcija je konstantna.

IZREK (Princip maksima):
Naj bo D (povezano) območje in $f\in\mathcal{H}(D)$. Denimo, da je f omejena na D. Tedaj je, ali

(1) $|f(z)|<\sup_D|f|$, $\forall z\in D$

ali

(2) f konstantna na D.

Dokaz:

Denimo, da obstaja $\alpha\in D$, da je $f(\alpha)=\sup_D|f|$. $(\Rightarrow)$ $|f(z)|$ konstantna $\forall z\in D$
$(f(z)\cdot\overline{f(z)}=A)$

Odvajamo:
$$\underbrace{\frac{\partial f}{\partial \bar z}}_{=0}\cdot\bar f + f\frac{\partial \bar f}{\partial \bar z}=0 \to f\cdot\bar{f}'=0 \quad \Big\}\ \frac{\partial}{\partial \bar z}$$
$$f'\cdot\bar{f}' + f\underbrace{\frac{\partial \bar f}{\partial z}}_{=0}=0 \to f'\cdot\bar{f}'=0 \quad \Big\}\ \frac{\partial}{\partial z}$$

$|f'|^2=0$
$\Rightarrow f'=0 \overset{D}{\Rightarrow} f=\text{konstantna}$

$A=\{z\in D;\ |f(z)|=\sup_D|f|\}$

Pokazali bomo:
- $A\neq\emptyset$
- A zaprta v D
- A odprta v D

$\Big\} \Rightarrow A=D$, ker D povezana
$\downarrow$
$|f(z)|=\sup_D|f|$, $\forall z\in D$

- $\alpha\in A$ ✓
- zaprtost v D: A je praslika točke $A=\sup_D|f|$ z zvezno funkcijo $z\mapsto|f(z)|$
  $\Rightarrow A$ zaprta v D
- LASTNOST POVPREČNE VREDNOSTI:

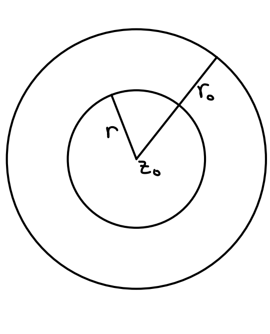
$z_0\in A \Rightarrow \exists r_0>0 \ni: \Delta(z_0,r_0)\subseteq A \Rightarrow z_0\in D^{odp} \Rightarrow \exists r_0>0 \ni: \Delta(z_0,r_0)\subseteq D$

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\varphi})\,d\varphi \Rightarrow |f(z_0)|\le\frac{1}{2\pi}\int_0^{2\pi}|f(z_0+re^{i\varphi})|\,d\varphi$$

$$\Rightarrow 0\le\frac{1}{2\pi}\int_0^{2\pi}\big[|f(z_0+re^{i\varphi})|-|f(z_0)|\big]\,d\varphi$$

$z_0\in A$: $|f(z_0)|=\sup_D|f|$ $\quad (|f(z_0+re^{i\varphi})|-|f(z_0)|\le 0)$

$\Rightarrow |f(z_0+re^{i\varphi})|=|f(z_0)|$ $\forall\varphi\in[0,2\pi]$



$\Delta(z_0,r)\subseteq A \quad \Leftarrow \begin{cases}\forall\varphi\in[0,2\pi]\\ \forall r\in(0,r_0)\end{cases}$

□

POSLEDICA: Naj bo D omejena odprta množica v $\mathbb{C}$ (ne nujno povezana).
Naj bo $f \in \mathcal{H}(D) \cap C(\overline{D})$.
Tedaj je: $\max\limits_{\overline{D}} |f| = \max\limits_{\partial D} |f|$.

Dokaz:
$|f| \in C(\overline{D})$, $\overline{D}$ kompaktna
$\Rightarrow \exists \max\limits_{\overline{D}} |f|$ in $\exists \max\limits_{\partial D} |f|$ $\quad \rightarrow$ očitno $\max\limits_{\overline{D}} |f| \geq \max\limits_{\partial D} |f|$

Denimo, da $\exists \alpha \in D$, da je $|f(\alpha)| = \max\limits_{\overline{D}} |f|$.

Naj bo D_0 povezana kompaktna množica, ki vsebuje α. D_0 je območje.
Po izreku je f konstantna na D_0.

$\Rightarrow |f(z)| = |f(\alpha)|$, $\forall z \in \overline{D}_0 \supseteq \partial D$ in $\partial D_0 \subseteq \partial D$. $\square$

IZREK (Princip identičnosti):
Naj bo D območje in $A \subseteq D$ s stekališčem v D.
($\alpha \in D$ je stekališče v $A \subseteq D$, če za $\forall \varepsilon > 0$. $\exists a \in A$, da je $a \in \Delta(\alpha, \varepsilon) \setminus \{\alpha\} = \Delta^*(\alpha, \varepsilon)$)
Naj bo $f \in \mathcal{H}(D)$, za katero velja $f(a) = 0$, $\forall a \in A$.
Tedaj je: $f \equiv 0$ na D.

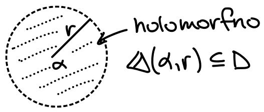
Dokaz:
$B = \{z \in D;\ f(z) = f'(z) = f''(z) = \dots = 0\}$

$$\left.\begin{array}{l} 1)\ B \text{ zaprta v } D \\ 2)\ B \text{ odprta v } D \\ 3)\ B \neq \emptyset \end{array}\right\} \overset{D \text{ območje}}{\Longrightarrow} B = D \Rightarrow f = 0 \text{ na } D$$

(1) $B = \bigcap\limits_{n=0}^{\infty} (f^{(n)})^{-1}(\{0\})$ (presek zaprtih množic je zaprta množica)
$f^{(n)}$ zvezni $\forall n$
$\Rightarrow B$ zaprta v D

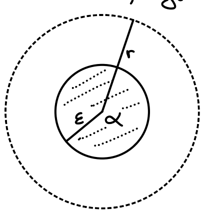
(2) $z_0 \in B$: $\exists r_0 > 0$. $\Delta(z_0, r_0) \subseteq D$:
$$f(z) = \sum_0^{\infty} \underbrace{\frac{f^{(n)}(z_0)}{n!}}_{0} (z - z_0)^n = 0 \text{ na } \Delta(z_0, r_0) \subseteq B.$$

(3) $A \subseteq D$, $\alpha \in D$ stekališče A



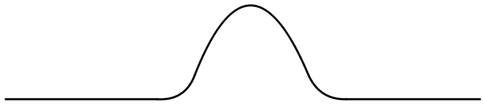
Uporabimo pomožno trditev:
ali: 2) $f = 0$ na $\Delta(\alpha, r) \Rightarrow \alpha \in B \neq \emptyset$
ali: 1) $f(z) = (z - \alpha)^N g(z)$, $g(z) \in \mathcal{H}(\Delta(\alpha, r))$ in $g(\alpha) \neq 0$

$\overset{1}{\rightarrow} \exists \varepsilon > 0$, $g(z) \neq 0$ za $z \in \Delta(\alpha, \varepsilon)$



Na $\Delta(\alpha, \varepsilon)$ ima f ničlo le v točki α, kar ni možno, če je α stekališče za A. $\times$ $\square$

Zgled:
$f(x) = x \sin \frac{1}{x}$



POSLEDICA: $A \subseteq D$ (kot prej). $f, g \in \mathcal{H}(D)$ in $f = g$ na A.
Tedaj je $f = g$ na D.

Dokaz:
$F = f - g$
$A \subseteq Z_F$ = ničle F
$\underset{\text{izrek}}{\Rightarrow}$ $F = 0$ na D oz. $f = g$ na D. $\square$

Zgled:
$e^{z+w} = e^z \cdot e^w$

1) $w \in \mathbb{R}$: $f(z) = e^{z+w}$
$g(z) = e^z \cdot e^w \in \mathcal{H}(\mathbb{C})$
$f = g$ na $\mathbb{R} \subseteq \mathbb{C}$
$\underset{\text{izrek}}{\Rightarrow}$ $f = g$ na $\mathbb{C}$

2) $z \in \mathbb{C}$: $f(w) = e^{z+w}$
$g(w) = e^z \cdot e^w \in \mathcal{H}(\mathbb{C})$
$f = g$ na $\mathbb{R} \subseteq \mathbb{R}$
$\underset{\text{izrek}}{\Rightarrow}$ $f = g$ na $\mathbb{C}$

TRDITEV (pomožna): Naj bo $f \in \mathcal{H}(\Delta(\alpha, r))$ in ima f ničlo v α: $f(\alpha) = 0$.
Potem, ali
(1) obstaja tak $N \in \mathbb{N}$ in $g \in \mathcal{H}(\Delta(\alpha, r))$, $g(\alpha) \neq 0$, da $f(z) = (z-\alpha)^N g(z)$
ali
(2) $f \equiv 0$.

Dokaz:
$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$; konvergenčni polmer vsaj $r > 0$.

f ima ničlo v α: i) $\exists n \in \mathbb{N}: a_n \neq 0$
ii) $\forall n \in \mathbb{N}: a_n = 0 \rightarrow f \equiv 0$

$N = \min\{n \in \mathbb{N}, a_n \neq 0\}$; $a_N \neq 0$

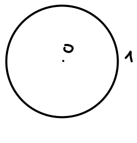
$$\begin{aligned} f(z) &= a_N (z-\alpha)^N + a_{N+1} (z-\alpha)^{N+1} + \cdots \\ &= (z-\alpha)^N (a_N + a_{N+1}(z-\alpha) + \cdots) \\ &= (z-\alpha)^N g(z) \end{aligned}$$

konvergenčni polmer vsaj $r > 0$.

$g(z)$: $g \in \mathcal{H}(\Delta(\alpha, r))$
$g(\alpha) = a_N \neq 0$ $\square$

OPOMBA: Če $f \neq 0$, je N = red ničle funkcije f v točki α.

Zgled:
$\sin \frac{1}{z-1} \in \mathcal{H}(\mathbb{C} \setminus \{1\})$ $\rightarrow$ ničle: $\frac{1}{z-1} = k\pi$; $k \in \mathbb{Z}$
$k \neq 0$: $\frac{1}{k\pi} + 1 = z$
$k < 0$

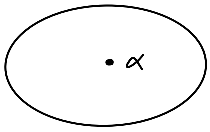


IZOLIRANJE SINGULARNE TOČKE

$\alpha \in D^{odp} \subseteq \mathbb{C}$

$D^* = D \setminus \{\alpha\}$ ... prebodena okolica α

$f \in \mathcal{H}(D^*) = \mathcal{H}(D \setminus \{\alpha\})$



Rečemo, da ima f izolirano singularno točko v α.

Zgledi:

(1) $f(z) = \frac{1}{z^{10}} \in \mathcal{H}(\mathbb{C} \setminus \{0\})$

(2) $f \in \mathcal{H}(D)$

$\forall \alpha \in D$ izolirana singularna točka za f

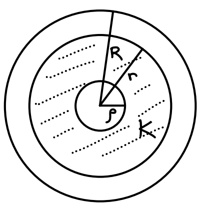
$\frac{\sin z}{z} \in \mathcal{H}(\mathbb{C} \setminus \{0\})$ (odpravljiva singularna točka)

$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

(3) $e^{\frac{1}{z}} \in \mathcal{H}(\mathbb{C} \setminus \{0\})$ (bistvena singularna točka)

$e^{\frac{1}{z}} = w \rightarrow \frac{1}{z} = \log w + 2k\pi i \rightarrow z_k = \frac{1}{\log|w| + \arg w \cdot i + 2k\pi i}$

$\alpha \in \mathbb{C}$; $\Delta(\alpha, R)$, $\Delta^*(\alpha, R)$



$K^{kompakt}$

$\exists\, 0 < \rho < r < R: \quad K \subseteq \Delta(\alpha, r) \setminus \Delta(\alpha, \rho)$

$z \in \Delta(\alpha, r) \setminus \overline{\Delta(\alpha, \rho)}$

Cauchyjeva formula: $f(z) = \frac{1}{2\pi i} \int\limits_{\partial(\Delta(\alpha,r) \setminus \overline{\Delta(\alpha,\rho)})} \frac{f(\xi)}{\xi - z} d\xi = \underbrace{\frac{1}{2\pi i} \int\limits_{|\xi - \alpha| = r} \frac{f(\xi)}{\xi - z} d\xi}_{\mathrm{I}} - \underbrace{\frac{1}{2\pi i} \int\limits_{|\xi - \alpha| = \rho} \frac{f(\xi)}{\xi - z} d\xi}_{\mathrm{II}}$

I) $\rho < |z - \alpha| < r$, $|\xi - \alpha| = r$:

$\frac{1}{\xi - z} = \frac{1}{(\xi - \alpha) - (z - \alpha)} = \frac{1}{(\xi - \alpha)\left(1 - \frac{z - \alpha}{\xi - \alpha}\right)} = \quad \left|\frac{z-\alpha}{\xi - \alpha}\right| \leq \frac{r'}{r} < 1$

$= \sum_0^\infty \frac{(z - \alpha)^n}{(\xi - \alpha)^{n+1}}$ $\quad \rho < \rho' \leq |z - \alpha| \leq r' < r$; $|\xi - \alpha| = r$

tu je konvergenca enakomerna

enakomerno konvergira

$\Rightarrow \frac{1}{2\pi i} \int\limits_{|\xi - \alpha| = r} \frac{f(\xi)}{\xi - z} d\xi = \sum_0^\infty \left(\frac{1}{2\pi i} \int\limits_{|\xi - \alpha| = r} \frac{f(\xi)}{(\xi - \alpha)^{n+1}} d\xi \right) (z - \alpha)^n$ ← Ta potenčna vrsta konvergira absolutno za $\forall z \in \Delta(\alpha, R)$ in enakomerno na kompaktih v $\Delta(\alpha, R)$

II) $\rho < \rho' \leq |z - \alpha|$, $|\xi - \alpha| = \rho$:

$-\frac{1}{\xi - z} = -\frac{1}{(\xi - \alpha) - (z - \alpha)} = \frac{1}{(z - \alpha)\left(1 - \frac{\xi - \alpha}{z - \alpha}\right)} = \quad \left|\frac{\xi - \alpha}{z - \alpha}\right| \leq \frac{\rho}{\rho'} < 1$

enakomerno

$= \sum_0^\infty \frac{(\xi - \alpha)^n}{(z - \alpha)^{n+1}}$

$\Rightarrow \frac{-1}{2\pi i} \int\limits_{|\xi - \alpha| = \rho} \frac{f(\xi)}{\xi - z} d\xi = \sum_0^\infty \left(\frac{1}{2\pi i} \int\limits_{|\xi - \alpha| = \rho} f(\xi)(\xi - \alpha)^n d\xi \right) (z - \alpha)^{-n-1}$ ← konvergira absolutno za $\forall z$, $|z - \alpha| > 0$ $(z \neq \alpha)$ in enakomerno na kompaktnih podmnožicah $\mathbb{C} \setminus \{\alpha\}$

$-n-1 = m$ negativno celo število

$\frac{1}{2\pi i} \int\limits_{|\xi - \alpha| = \rho} f(\xi)(\xi - \alpha)^{-m-1} d\xi = \frac{1}{2\pi i} \int\limits_{|\xi - \alpha| = \rho} \frac{f(\xi)}{(\xi - \alpha)^{m+1}} d\xi$

$$\Rightarrow f(z)=\sum_{n\in\mathbb{Z}} a_n(z-\alpha)^n = \underbrace{\sum_{n=0}^{\infty} a_n(z-\alpha)^n}_{\substack{\Delta(\alpha,R)\\ \text{REGULARNI DEL}}} + \underbrace{\sum_{-\infty}^{-1} a_n(z-\alpha)^n}_{\substack{\mathbb{C}\setminus\{\alpha\}\\ \text{GLAVNI DEL}}}$$

$\leftarrow$ LAURENTOV RAZVOJ funkcije f v okolici izolirane singularne točke
(LAURENTOVA VRSTA f v okolici α)

Zgledi:

1) $f(z)=\frac{1}{z}=z^{-1}$ $\rightarrow$ regularni del: 0
glavni del: $\frac{1}{z}$ (pol)

2) $f(z)=e^{\frac{1}{z}}=\underbrace{1}_{\text{regularni del}}+\underbrace{\frac{1}{z}+\frac{1}{2!\,z^2}+\frac{1}{3!\,z^3}+\dots\frac{1}{n!\,z^n}+\dots}_{\text{glavni del}}$ (bistvena singularnost)

3) $f(z)=\frac{\sin z}{z}$ $\rightarrow$ glavni del: 0
regularni del: $\frac{\sin z}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\dots$ (odpravljiva singularnost)

IZREK (razvoj v Laurentovo vrsto):
Naj bo $f\in\mathcal{H}(\Delta^*(\alpha,R))$.
Tedaj se f na $\Delta^*(\alpha,R)$ da razviti v Laurentovo vrsto $f(z)=\sum_{-\infty}^{\infty} a_n(z-\alpha)^n$.
Vrsta konvergira absolutno na $\Delta^*(\alpha,R)$ in enakomerno na kompaktnih podmnožicah.
Za vsak $0<r<R$, $\forall n\in\mathbb{Z}$, je:
$$a_n=\frac{1}{2\pi i}\int_{|\xi-\alpha|=r}\frac{f(\xi)}{(\xi-\alpha)^{n+1}}\,d\xi .$$
1) Pri tem je regularni del vrste $\sum_0^{\infty} a_n(z-\alpha)^n$, ki konvergira absolutno na $\Delta(\alpha,R)$ in enakomerno na kompaktnih podmnožicah.
2) Glavni del $\sum_{-\infty}^{-1} a_n(z-\alpha)^n$ konvergira absolutno na $\mathbb{C}\setminus\{\alpha\}$ in enakomerno na kompaktnih podmnožicah.

DEFINICIJA: Naj bo α izolirana singularna točka za f $(\in\mathcal{H}(\Delta^*(\alpha,R)))$.
(1) Točka α je ODPRAVLJIVA SINGULARNA TOČKA za f oz. f ima v α ODPRAVLJIVO SINGULARNOST, če $a_{-n}=0$, $\forall n\in\mathbb{N}$.
V tem primeru je
$$\tilde{f}(z)=\begin{cases} f(z); & z\in\Delta^*(\alpha,R)\\ a_0; & z=\alpha\end{cases};\quad \tilde{f}\in\mathcal{H}(\Delta(\alpha,R))$$
f se da holomorfno razširiti na $\Delta(\alpha,R)$.
(2) f ima v α POL oz. α je POL STOPNJE $n\in\mathbb{N}$ za f, če: $a_{-n}\neq 0$ in $a_{-N-n}=0$, $n\in\mathbb{N}$.
$$f(z)=\sum_0^{\infty} a_n(z-\alpha)^n+\frac{a_{-1}}{z-\alpha}+\frac{a_{-2}}{(z-\alpha)^2}+\dots+\frac{a_{-N}}{(z-\alpha)^N}+\dots$$
Glavni del ima končno mnogo neničelnih členov.
(3) f ima v α BISTVENO SINGULARNOST oz. α je BISTVENA SINGULARNA TOČKA za f, če je $a_{-n}\neq 0$ za neskončno mnogo $n\in\mathbb{N}$.

Te tri singularnosti karakteriziramo glede na obnašanje f v okolici $\Delta^*(\alpha,R)$.

IZREK: Naj ima $f\in\mathcal{H}(\Delta^*(\alpha,R))$ v α izolirano singularno točko.
Tedaj je α odpravljiva singularna točka natanko tedaj, ko $\exists\, 0<r<R$, da je f omejena na $\Delta^*(\alpha,r)$.

Dokaz:
($\Rightarrow$): Naj ima f v α odpravljivo singularnost.
Potem $f=\tilde{f}$ na $\Delta^*(\alpha,R)$ $\rightarrow$ $\tilde{f}$ holomorfna na $\Delta(\alpha,R)$ in $\tilde{f}(z)=\sum_0^{\infty} a_n(z-\alpha)^n$
$\Rightarrow$ $|\tilde{f}|$ omejena na $\overline{\Delta(\alpha,r)}$, $\forall 0<r<R$. $\Rightarrow$ f omejena na $\Delta^*(\alpha,r)$.

($\Leftarrow$): Naj bo f omejena na $\Delta^*(\alpha, r)$, za nek $0<r<R$.

$n \in \mathbb{N}$:

$$a_{-n} = \frac{1}{2\pi i}\int_{|\xi-\alpha|=r'} \frac{f(\xi)}{(\xi-\alpha)^{-n+1}}\,d\xi \quad ; \quad 0<r'<r$$

$$= \frac{1}{2\pi i}\int_{|\xi-\alpha|=r'} f(\xi)(\xi-\alpha)^{n-1}\,d\xi .$$

$\exists M<\infty : |f(z)|<M$, $\forall z \in \Delta^*(\alpha, r)$

$$\rightarrow |a_{-n}| \leq \frac{1}{2\pi i}\int_{|\xi-\alpha|=r'} M|\xi-\alpha|^{n-1}|d\xi| = M r'^{n-1} r' = M r'^{n} \xrightarrow{r'\to 0} \quad a_{-n}=0$$

$\Rightarrow \alpha$ odpravljiva singularna točka $\square$

TRDITEV: Naj bo $f \in \mathcal{H}(\Delta^*(\alpha,R))$.
pomožna
Funkcija f ima v α pol stopnje N natanko takrat, ko: $\exists g \in \mathcal{H}(\Delta(\alpha,R))$, $g(\alpha)\neq 0$ in $f(z) = \frac{g(z)}{(z-\alpha)^N}$, $\forall z \in \Delta^*(\alpha,R)$.

Dokaz:

($\Rightarrow$): f ima v α pol stopnje N:

$$\rightarrow f(z) = \frac{a_{-N}}{(z-\alpha)^N} + \dots + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \dots =$$

$$= \frac{a_{-N} + a_{-N+1}(z-\alpha) + \dots + a_{-1}(z-\alpha)^{N-1} + a_0(z-\alpha)^N + \dots}{(z-\alpha)^N} =$$

$$= \frac{g(z)}{(z-\alpha)^N}$$ $\leftarrow$ konvergenčni polmer je vsak R
$g \in \mathcal{H}(\Delta(\alpha,R))$
$g \neq 0$

($\Leftarrow$): Denimo: $f(z) = \frac{g(z)}{(z-\alpha)^N}$; $g(\alpha)\neq 0$, $g \in \mathcal{H}(\Delta(\alpha,R))$

g razvijemo v potenčno vrsto: $= \frac{b_0 + b_1(z-\alpha) + \dots + b_n(z-\alpha)^N + \dots}{(z-\alpha)^N} =$

$$= \frac{\overset{\neq 0}{b_0}}{(z-\alpha)^N} + \frac{b_1}{(z-\alpha)^{N-1}} + \dots$$

$\Rightarrow f$ ima v α pol stopnje N $\square$

IZREK: Naj ima $f \in \mathcal{H}(\Delta(\alpha,R))$ v α izolirano singularno točko.
Tedaj ima f v α pol natanko takrat, ko je $\lim_{z\to\alpha} |f(z)| = \infty$.

Dokaz:

($\Rightarrow$): Naj ima f v α pol.
Tedaj obstaja $N \in \mathbb{N}$, $g \in \mathcal{H}(\Delta(\alpha,R))$, $g(\alpha)\neq 0$.

$$f(z) = \frac{g(z)}{(z-\alpha)^N}$$

$$\lim_{z\to\alpha}|f(z)| = \lim_{z\to\alpha}\frac{|g(z)|}{|z-\alpha|^N} = \infty \qquad \lim_{z\to\alpha}|g(z)| = |g(\alpha)| \neq 0$$

($\Leftarrow$): Denimo, da je $\lim_{z\to\alpha}|f(z)| = \infty$

Za vsak $M \in (0,\infty)$ obstaja $\varepsilon>0$: $\forall z$: $0<|z-\alpha|<\varepsilon \Rightarrow |f(z)|>M$

$\Rightarrow f$ nima ničel na $\Delta^*(\alpha,\varepsilon)$

Definiramo $F(z) = \frac{1}{f(z)}$ na $\Delta^*(\alpha,\varepsilon)$

F ima izolirano odpravljivo točko v α $\quad \lim_{z\to\alpha} |F(z)| = 0$

$\Rightarrow$ F je omejena v neki punktirani okolici $\Delta^*(\alpha, \varepsilon')$, $\varepsilon \geq \varepsilon' > 0$

$\Rightarrow$ F ima v α odpravljivo singularnost z vrednostjo 0

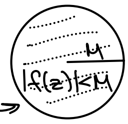
Torej: $F(z) = (z-\alpha)^N h(z)$, $n \in \mathbb{N}$, $h \in \mathcal{H}(\Delta(\alpha, \varepsilon'))$, $h(\alpha) \neq 0$

$\Rightarrow f(z) = \frac{\frac{1}{h(z)}}{(z-\alpha)^N}$; $g(z) = \frac{1}{h(z)}$, $g(\alpha) \neq 0$ $\square$

IZREK: Naj bo $f \in \mathcal{H}(\Delta^*(\alpha, R))$ oz. f ima v α izolirano singularnost.
f ima v α bistveno singularnost natanko takrat, ko za vsak dovolj majhen $0<r<R$ velja $\overline{f(\Delta^*(\alpha, r))} = \mathbb{C}$.
(slika $\Delta^*(\alpha, r)$ z f je gosta v $\mathbb{C}$)

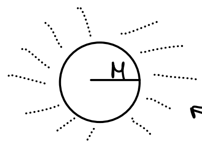
Ekvivalentno: f v α nima bistvene singularnosti $\Leftrightarrow \exists r$, $0<r<R$, da $f(\Delta^*(\alpha,r))$ ni gosta v $\mathbb{C}$

Dokaz:
($\Rightarrow$): f v α nima bistvene singularnosti
Torej ima f v α ali odpravljivo singularnost ali pol
Denimo, da ima f v α odpravljivo singularnost.
$\Leftrightarrow$ f je omejena v neki preboden okolici α: $\Delta^*(\alpha, r)$



slika ni gosta v $\mathbb{C}$

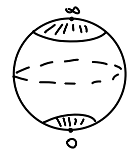
Denimo, da ima f v α pol.
$\Leftrightarrow \lim_{z\to\alpha} |f(z)| = \infty$



$\exists 0<r<R$, da je $|f(z)| \geq M > 0$

slika ni gosta v C

($\Leftarrow$): Denimo, da za nek $0<r<R$ slika $f(\Delta^*(\alpha,r))$ ni gosta v $\mathbb{C}$.
$\Rightarrow \exists A \in \mathbb{C}$ in $\rho > 0$, da $f(\Delta^*(\alpha,r)) \cap \Delta(A, \rho) = \emptyset$
$\forall z \in \Delta^*(\alpha, r): |f(z) - A| \geq \rho > 0$.



Naj bo:
$$h(z) = \frac{1}{f(z)-A} \in \mathcal{H}(\Delta^*(\alpha,r)) \; ; \quad |h(z)| = \frac{1}{|f(z)-A|} \leq \frac{1}{\rho}$$
h ima odpravljivo singularnost v α
$\Rightarrow h(z) = (z-\alpha)^N g(z)$; kjer $N \in \mathbb{N} \cup \{0\}$, $g \in \mathcal{H}(\Delta(\alpha, r))$, $g(\alpha) \neq 0$
$\frac{1}{f(z)-A} = (z-\alpha)^N g(z)$ $\qquad g(z) \neq 0$ na $\Delta(\alpha, r')$
$\frac{\frac{1}{g(z)}}{(z-\alpha)^N} + A = f(z)$, $z \in \Delta^*(\alpha, r')$, $0<r'<r$
$N = 0 \Rightarrow$ f ima v α odpravljivo singularnost
$N \in \mathbb{N} \Rightarrow$ f ima v α pol (stopnje N) $\square$

IZREK (Veliki Picarrov izrek):
Naj bo $f \in \mathcal{H}(\Delta^*(\alpha, R))$.
Funkcija f ima v α bistveno singularnost natanko tedaj, ko za vsak $0<r\leq R$ f zavzame na $\Delta^*(\alpha, r)$ vse vrednosti v $\mathbb{C}$, z izjemo morda ene, neskončno mnogokrat.

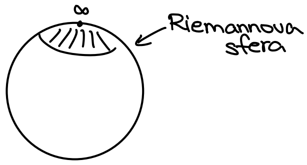
OPOMBA: $f(\Delta^*(\alpha, r)) \to \mathbb{C}$
$\qquad\qquad \searrow \mathbb{C} \setminus \{A\}$

Zgled:
∘) $f(z) = e^{\frac{1}{z}}$:
$f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$

IZREK (Mali Picarrov izrek):

Naj bo f nekonstantna cela holomorfna funkcija: $f \in \mathcal{H}(\mathbb{C})$.
Tedaj f zavzame vse vrednosti v $\mathbb{C}$, razen morda ene.

Dokaz: (veliki Picarrov izrek $\Rightarrow$ mali Picarrov izrek)



Točka ∞ je izolirana singularna točka za f
$g(z) = f(\frac{1}{z})$, $g \in \mathcal{H}(\mathbb{C} \setminus \{0\})$
0 izolirana singularna točka za g

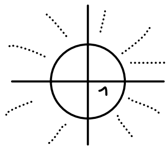
1) 0 odpravljiva singularna točka za g: $\exists r>0$ in $M<\infty \ni: |g(z)| \leq M$, $\forall z \in \Delta^*(0,r)$
$|f(\frac{1}{z})| \leq M \Leftrightarrow \frac{1}{r} < \frac{1}{|z|} < \infty$
$\frac{1}{z} = w$

za $|w| \leq \frac{1}{r}$. $\exists M' \ni: |f(w)| \leq M'$ $\rightarrow$ zveznost
za $\forall w$: $\frac{1}{r} < |w|$ je $|f(w)| \leq M$

$\Rightarrow f$ omejena $\overset{\text{Liouvill}}{\Rightarrow}$ f konstantna

Ni možno glede na predpostavke.

2) 0 je pol stopnje $N \in \mathbb{N}$ za g
$\exists h \in \mathcal{H}(\Delta(0,2))$; $h(0) \neq 0$ in $g(z) = \frac{h(z)}{z^N}$
$\Rightarrow \exists M \ni: |h(z)| \leq M$, $z \in \overline{\Delta(0,1)}$
$\Rightarrow |g(z)| \leq \frac{M}{|z|^N}$ za $\forall |z| \leq 1$, $z \neq 0$ $\rightarrow |f(\frac{1}{z})| \leq \frac{M}{|z|^N}$, $w = \frac{1}{z}$, $|w| \geq 1$
$\Rightarrow |f(w)| \leq M|w|^N$



Na $\overline{\Delta(0,1)}$ je f omejena.
Torej $\exists M' < \infty \ni: \forall w.\ |f(w)| \leq M'(1+|w|^N)$
$\Rightarrow f$ nekonstanten polinom
$\Rightarrow \forall A \in \mathbb{C}.\ \exists$ rešitev enačbe $f(w) = A$ (osnovni izrek algebre)
$\Rightarrow f(\mathbb{C}) = \mathbb{C}$

3) 0 je bistvena singularna točka za g
$r>0$: $g(\Delta^*(0,r)) \rightarrow \mathbb{C}$
$f(\mathbb{C} \setminus \overline{\Delta(0,\frac{1}{r})}) \rightarrow \mathbb{C} \setminus \{A\}$ $|z| < r$, $w = \frac{1}{z}$, $|w| > \frac{1}{r}$ $\square$

POSLEDICA: Vsaka cela holomorfna funkcija, ki ne zavzame dveh vrednosti, je konstantna.

Zgled:
$f, g \in \mathcal{H}(\mathbb{C})$: $e^f + e^g = 1$ na $\mathbb{C}$
ne zavzameta 0 in 1 $\Rightarrow e^f, e^g$ konstantni
$\Rightarrow f, g$ konstantni funkciji

RAZVOJ f V LAURENTOVO VRSTO V OKOLICI ∞:

Naj bo $f \in \mathcal{H}(\mathbb{C} \setminus \overline{\Delta(0,R)})$ (prebodena okolica ∞)

Oglejmo si $g(z) = f(\frac{1}{z})$, $g \in \mathcal{H}(\Delta^*(0, \frac{1}{R}))$
0 izolirana singularna točka za g
$0 < |R| < \frac{1}{R}$
razvoj: $g(z) = \sum_{-\infty}^{\infty} a_n z^n = \sum_{-\infty}^{-1} a_n z^n + \sum_{0}^{\infty} a_n z^n$; $n \in \mathbb{Z}$

$f(w) = g(\frac{1}{w}) = g(z)$

$$f(w) = \underbrace{\sum_{-\infty}^{-1} a_n w^{-n}}_{\text{glavni del}} + \underbrace{\sum_{0}^{\infty} a_n w^{-n}}_{\text{regularni del}} \quad ; \ |w| > R$$

Tip singularnosti f v ∞ je tip singularnosti g v 0:

1) ∞ odpravljiva singularna točka za f: $f(w) = a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots$

2) ∞ pol stopnje $N \in \mathbb{N}$ za f: $f(w) = a_{-N} w^N + a_{-N+1} w^{N-1} + \cdots + a_{-1} w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots$

3) ∞ bistvena singularna točka za f: $a_{-n} \neq 0$ za neskončno mnogo $n \in \mathbb{N}$
$f(w) = \cdots + a_{-2} w^2 + a_{-1} w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots$ ima bistveno singularnost v ∞
$f(w) = e^w = 1 + \underbrace{w + \frac{w^2}{2} + \cdots + \frac{w^n}{n!} + \cdots}_{\text{glavni del v } \infty}$

OPOMBA: $f \cdot g = 0$ na D (območje), $f, g \in \mathcal{H}(D)$
$f \not\equiv 0$. $\exists \alpha \in D$. $f(\alpha) \neq 0$
$\Rightarrow \exists \Delta(\alpha, r)$, $f(z) \neq 0$
$\Rightarrow g \equiv 0$ na $\Delta(\alpha, r)$
princip identičnosti $\Rightarrow g \equiv 0$ na D

MEROMORFNE FUNKCIJE

D območje, $A \subseteq D$ diskretna podmnožica brez stekališča v D.
$f \in \mathcal{H}(D \setminus A)$ in v vsaki točki A ima f pol ali odpravljivo singularnost.
Tedaj rečemo, da je f MEROMORFNA na D.

OPOMBE: 1) A ni fiksna množica, je odvisna od f.

2) $f: D \to \mathbb{C}P^1$ ← Riemannova sfera
točke iz $A \mapsto \infty$

3) $f = \frac{g}{h}$, $g, h \in \mathcal{H}(D)$, $h \not\equiv 0 \Rightarrow f$ meromorfna na D
$(z-\alpha)^N H(z) = h(z)$ v okolici α $\rightarrow \frac{g(z)}{h(z)} = \frac{1}{(z-\alpha)^N} \frac{g(z)}{H(z)}$

OPOMBA: Vsaka meromorfna funkcija na D je take oblike.

f meromorfna na D: $f: D \to \hat{\mathbb{C}} = \mathbb{C}P^1$

TRDITEV: Naj bo D območje in f meromorfna funkcija na D.
Če ima množica ničel f stekališče v D, je $f \equiv 0$.
(oz. če $f \not\equiv 0$, množica ničel nima stekališča v D)

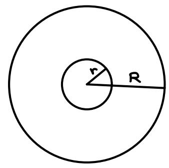
Dokaz:
Naj bo Z množica ničel f na D. Velja $Z \subseteq D \setminus A$ ← množica polov f
Ker je D povezana, je tudi $D \setminus A$ povezana.
Denimo, da $f \not\equiv 0 \Rightarrow$ množica Z nima stekališč v $D \setminus A$
Možna stekališča so na $\partial D \setminus A = \partial D \cup A$.

Ker je A množica polov za f, funkcija f nima ničel v okolici vsake točke $\alpha \in A$.
Možno stekališče Z je na ∂D. A nima stekališča v D. $\square$

OPOMBA: Če je f meromorfna na D (območje), $f \not\equiv 0$. $\Rightarrow \frac{1}{f}$ meromorfna na D.
ničle $f \sim$ poli $\frac{1}{f}$ in poli $f \sim$ ničle $\frac{1}{f}$.

Naj bo $f \in \mathcal{H}(\Delta^*(\alpha, R))$, $0<r<R$ in $f(z) = \sum_{-\infty}^{\infty} a_n (z-\alpha)^n$.

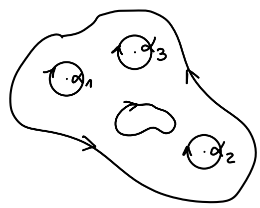
$$\frac{1}{2\pi i} \int_{|z-\alpha|=r} f(z)\,dz = a_{-1} = \mathrm{Res}(f, \alpha) \leftarrow \text{RESIDUUM } f \text{ v } \alpha$$



IZREK (izrek o ostankih oz. izrek o residuih)
Naj bo D omejena odprta množica v $\mathbb{C}$; z odprtim gladkim robom, sestavljenim iz končnega števila odsekoma gladkih sklenjenih krivulj, orientiranih pozitivno glede na D.
Naj bodo $\alpha_1, \dots, \alpha_N \in D$ in $f \in \mathcal{H}(D \setminus \{\alpha_1, \dots, \alpha_N\}) \cap C^1(\overline{D} \setminus \{\alpha_1, \dots, \alpha_N\})$.
Tedaj:

$$\frac{1}{2\pi i} \int_{\partial D} f(z)\,dz = \sum_{j=1}^{N} \mathrm{Res}(f, \alpha_j)$$

Dokaz:
Naj bo $r>0$, da velja: 1) $\overline{\Delta(\alpha_j, r)} \subseteq D$, $\forall j$
2) $\overline{\Delta(\alpha_j, r)} \cap \overline{\Delta(\alpha_\ell, r)} = \emptyset$, $j \neq \ell$



$D_r = D \setminus \bigcup_1^N \overline{\Delta(\alpha_j, r)}$
$f \in \mathcal{H}(D_r) \cap C^1(\overline{D}_r)$

Cauchyjev izrek: $\frac{1}{2\pi i} \int_{\partial D_r} f(z)\,dz = 0$; $\partial D_r = \partial D \cup \bigcup_1^N \partial \Delta(\alpha_j, r)$

Upoštevajoč orientacijo:

$$0 = \frac{1}{2\pi i} \int_{\partial D} f(z)\,dz - \sum_1^N \overbrace{\frac{1}{2\pi i} \int_{\partial \Delta(\alpha_j, r)} f(z)\,dz}^{\mathrm{Res}(f, \alpha_j)}$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\partial D} f(z)\,dz = \sum_1^N \mathrm{Res}(f, \alpha_j)$$

$\square$

OPOMBA: Pogosto: $D \subseteq \overline{D} \subseteq \Omega$, f meromorfna na Ω s končno mnogo poli v D in brez polov na ∂D.

TRDITEV: Naj bo $f \in \mathcal{H}(\Delta^*(\alpha, R))$ in naj ima f v α pol stopnje N.
Tedaj:

$$\mathrm{Res}(f, \alpha) = \frac{1}{(N-1)!} \lim_{z \to \alpha} \frac{d^{N-1}}{dz^{N-1}} \left[(z-\alpha)^N f(z) \right].$$

Dokaz:

$$f(z) = \frac{a_{-N}}{(z-\alpha)^N} + \dots + \frac{a_{-1}}{(z-\alpha)} + a_0 + \dots$$

$$\Rightarrow (z-\alpha)^N f(z) = a_{-N} + \dots + a_{-1}(z-\alpha)^{N-1} + a_0 (z-\alpha)^N + \dots$$

$$\frac{d^{N-1}}{dz^{N-1}} (z-\alpha)^N f(z) = (N-1)!\, a_{-1} + N!\, a_0 (z-\alpha) + \dots$$

$\square$

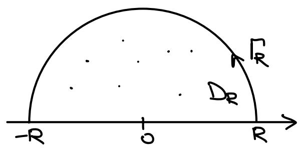
POSLEDICA: $N=1$: $\mathrm{Res}(f, \alpha) = \lim_{z \to \alpha} \left[(z-\alpha) f(z) \right]$.

IZREK: Naj bosta p in q polinoma in naj velja: $q(x) \neq 0$, $\forall x \in \mathbb{R}$ in $\deg p + 2 \leq \deg q$.
Tedaj: $\displaystyle\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{j=1}^{N} \operatorname{Res}\left(\frac{p}{q}, \alpha_j\right)$, kjer so $\alpha_1, \dots, \alpha_N$ poli $\frac{p}{q}$ v zgornji polravnini.

Dokaz:
$f(z) = \frac{p(z)}{q(z)}$

Predpostavimo: $\displaystyle\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{p(x)}{q(x)}\, dx$ dostaja.

$D_R = \{ z \in \mathbb{C};\ |z| < R,\ \operatorname{Im} z > 0 \}$



Naj bodo $\alpha_1, \dots, \alpha_N$ poli $\frac{p}{q}$ v $\operatorname{Im} z > 0$.
$\exists R > 0 \ni: |\alpha_j| < R\ \forall j = 1, \dots, N$ (za vse take R je integral iste vrednosti)

Po izreku o ostankih: $\displaystyle\underbrace{\int_{-R}^{R} f(z)\, dz}_{\downarrow R\to\infty\ \int_{-\infty}^{\infty}\frac{p(x)}{q(x)}dx} + \int_{\Gamma_R} f(z)\, dz = \int_{\partial D_R} f(z)\, dz = 2\pi i \sum_1^N \operatorname{Res}(f, \alpha_j)$

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz = 0$$

Parametrizacija: $z = Re^{i\varphi}$, $\varphi \in [0, \pi]$: $\displaystyle\left|\int_{\Gamma_R} f(z) dz\right| = \left|\int_0^{\pi} f(Re^{i\varphi}) Rie^{i\varphi}\, d\varphi\right| \leq R \int_0^{\pi} |f(Re^{i\varphi})|\, d\varphi \xrightarrow{R\to\infty} 0$

$f(z) = \frac{p(z)}{q(z)}$
$|Rf(Re^{i\varphi})| = R \frac{|p(Re^{i\varphi})|}{|q(Re^{i\varphi})|} \xrightarrow{R\to\infty} 0$, $\deg p + 2 < \deg q$

$\square$

Zgled: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$

$= 2\pi i \cdot \operatorname{Res}\left(\frac{1}{1+z^2}, i\right)$
$= 2\pi i \cdot \lim_{z\to i} \left(\frac{z-i}{1+z^2}\right) =$
$= 2\pi i \lim_{z\to i} \frac{1}{2z} =$
$= \frac{2\pi i}{2i} =$
$= \pi$

$p(z) = 1$
$q(z) = 1 + z^2$; ničle q: $\pm i$
i je edini pol v $\operatorname{Im} z > 0$
(1. stopnje)

Zgled: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i \cdot \left[\operatorname{Res}\left(\frac{1}{1+z^4}, \alpha_1\right) + \operatorname{Res}\left(\frac{1}{1+z^4}, \alpha_2\right)\right] = 2\pi i \cdot \left[\lim_{z\to\alpha_1} \frac{z-\alpha_1}{1+z^4} + \lim_{z\to\alpha_2} \frac{z-\alpha_2}{1+z^4}\right] =$

$p(z) = 1$
$q(z) = 1 + z^4 \to$ poli: $1 + z^4 = 0 \to z = e^{i\frac{\pi}{4}}$

$= 2\pi i \cdot \left[\lim_{z\to\alpha_1} \frac{1}{4z^3} + \lim_{z\to\alpha_1} \frac{1}{4z^3}\right] =$
$= 2\pi i \left[\frac{1}{4\alpha_1^3} + \frac{1}{4\alpha_2^3}\right] = \frac{\pi i}{2}\left[\alpha_1^{-3} + \alpha_2^{-3}\right] =$
$= \frac{\pi i}{2}\left[e^{-i\frac{3\pi}{4}} + e^{-i\frac{\pi}{4}}\right] = \frac{\pi i}{2}\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right] =$
$= \frac{\pi}{\sqrt{2}}$

POSLEDICE IZREKA O OSTANKIH

(*) $\begin{cases} D \subseteq \overline{D} \subseteq \Omega;\ D, \Omega \text{ odprti množici; } D \text{ omejena} \\ f \text{ meromorfna na } \Omega \\ \partial D \text{ odsekoma gladek, sestavljen iz končnega števila odsekoma gladkih} \\ \quad \text{sklenjenih krivulj, orientiranih skladno z } D. \end{cases}$

IZREK (princip argumenta):

Denimo, da f nima polov in ničel na ∂D. in velja (*)

Tedaj:
$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}dz = N_f - P_f \ ; \text{ kjer } \begin{array}{l} N_f = \# \text{ ničel } f \text{ na } D \text{ štetih s kratnostjo} \\ P_f = \# \text{ polov } f \text{ na } D \text{ štetih s kratnostjo}\end{array}$$

Dokaz:

singularnosti $\frac{f'}{f}$ so v ničlah in polih f

α ničla ali pol f v D:

$$\rightarrow f(z) = (z-\alpha)^N g(z), \quad N \in \mathbb{Z}\setminus\{0\}, \ g \in \mathcal{H}(\Delta(\alpha,R)), \ g(\alpha)\neq 0$$

$$f'(z) = N(z-\alpha)^{N-1}g(z) + (z-\alpha)^N g'(z)$$

$$\rightarrow \left(\frac{f'}{f}\right)(z) = N\frac{1}{z-\alpha} + \frac{g'(z)}{g(z)}$$

$$\Rightarrow \operatorname{Res}\left(\frac{f'}{f}, \alpha\right) = N$$

Po izreku o ostankih je: $\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}dz = N_f - P_f$ $\square$

OPOMBE: 1) f ima na D končno število ničel in polov

2) $\frac{f'}{f} = (\log f)' = (\underbrace{\log|f|}_{\text{dobro definirano}} + i\arg f)'$

IZREK (Rouchéjev izrek):

Naj bosta D in Ω kot v (*). Naj bo $f: [0,1]_t \times \Omega_z \to \mathbb{C}$ zvezna funkcija, ki je za vsak $t\in[0,1]$ holomorfna v z.

Denimo, da $f(t,z)\neq 0$ $\forall t\in[0,1]$ in $\forall z \in \partial D$.

Tedaj je število ničel $f(0,\cdot)$ enako številu ničel $f(1,\cdot)$.

Dokaz:
$$F(t) = \frac{1}{2\pi i}\int_{\partial D}\frac{f'(t,z)}{f(t,z)}dz = \# \text{ ničel } f(t,\cdot) \text{ na } D \in \mathbb{N}\cup\{0\}$$

integral s parametrom; zvezno v t

$\Rightarrow$ F zvezna na $[0,t]$ + celoštevilske vrednosti

$\Rightarrow$ F konstantna $\square$

Klasična oblika Rouchéjevega izreka:
1) D, Ω kot v (*)
2) $f, g \in \mathcal{H}(\Omega)$
3) $|g(z)| < f(z)$ na ∂D

Tedaj imata f in $f+g$ enako število ničel na D.

Dokaz:

$t\in[0,1]$: $f(t,z) = f(z) + tg(z)$

∂D: $|f(z)| > t|g(z)|$

$\forall t\in[0,1]$ in $f(t,z)$ nima ničel na ∂D

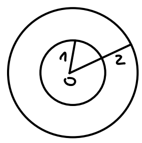
$f(0,z) = f(z)$ in $f(1,z) = (f+g)(z)$ $\square$

Zgled:

$p(z) = \underbrace{z^5}_{g} + \overbrace{3z+1}^{f}$ ima 5 ničel v $\mathbb{C}$

Koliko ničel je na: $1<|z|<2$?

$|z|=1$: $3=|3z|>\underbrace{|z^5|+1}_{2}\geq|z^5+1|$



na $|z|<1$ ima ta polinom 1 ničlo

$|z|=2$: $|z|^5=2^5=32>|3z|+1\geq|3z+1|$

$f(z)=z^5$, $g(z)=3z+1$

$p(z)$ ima na $|z|<2$ vseh 5 ničel, na $1<|z|<2$ pa ima $p(z)$ 4 ničle.

TRDITEV: Naj bo $\overline{\Delta(\alpha,r)}\subseteq D^{odp}\subseteq\mathbb{C}$ in $f\in\mathcal{H}(D)$ ter naj velja $|f(\alpha)|<\min\limits_{\partial\Delta(\alpha,r)}|f(z)|$. Tedaj ima f ničlo na $\Delta(\alpha,r)$.

Dokaz:

(1) Če f nima ničle na $\overline{\Delta(\alpha,r)}$, za $\frac{1}{f}$ v okolici $\overline{\Delta(\alpha,r)}$ velja princip maksima:

$$\max_{\overline{\Delta(\alpha,r)}}\frac{1}{|f|}=\max_{\partial\Delta(\alpha,r)}\frac{1}{|f|}\quad\text{in}\quad\frac{1}{\min\limits_{\overline{\Delta(\alpha,r)}}|f|}=\frac{1}{\min\limits_{\partial\Delta(\alpha,r)}|f|}\rightarrow\min_{\overline{\Delta(\alpha,r)}}|f|=\min_{\partial\Delta(\alpha,r)}|f|$$

Vemo: $\min\limits_{\overline{\Delta(\alpha,r)}}|f|\leq|f(\alpha)|\leq\min\limits_{\partial\Delta(\alpha,r)}|f|$

$\Rightarrow f$ ima ničlo na $\Delta(\alpha,r)$ $\square$

(2) Naj bosta $f(z)-f(\alpha)=f(z)+(\underset{g}{\underset{\parallel}{-f(\alpha)}})$ in $f(z)$.

Velja $|g(z)|\leq\min\limits_{\partial\Delta(\alpha,r)}|f|$.

Torej imata f in $f-f(\alpha)$ enako število ničel na $\Delta(\alpha,r)$.

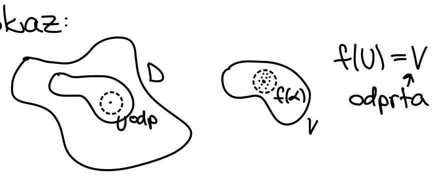
$z\mapsto f(z)-f(\alpha)$ ima ničlo v α in ima vsaj eno ničlo.

$\Rightarrow f$ ima vsaj eno ničlo na $\Delta(\alpha,r)$ $\square$

DEFINICIJA: Preslikava $f:(M,d)\to(N,\rho)$ je ODPRTA, če slika odprte množice iz (M,d) v odprte množice v (N,ρ).

IZREK: Naj bo D območje v $\mathbb{C}$ in $f:D\to\mathbb{C}$ nekonstantna holomorfna funkcija. Tedaj je f odprta preslikava.

Dokaz:



$f(U)=V$ odprta podmnožica $\mathbb{C}$

f je odprta preslikava, če za $\forall\alpha\in D$. $\exists r>0$, da je $f(\Delta(\alpha,r))\supseteq\Delta(f(\alpha),r')$, za nek $r'>0$.

Opazujemo: $f(z)-f(\alpha)=(z-\alpha)^N g(z)$ za $N\in\mathbb{N}$, $g\in\mathcal{H}(D)$, $g(\alpha)\neq0$.

(dovolj: g holomorfna v okolici $\overline{\Delta(\alpha,r)}$)

$g(z)\neq0$ na $\overline{\Delta(\alpha,r)}$: $|(z-\alpha)^N g(z)|=r^N|g(z)|\geq r^N\min\limits_{\partial\Delta(\alpha,r)}|g(z)|>0$; $|z-\alpha|=r$

$w\in\Delta(f(\alpha),\underbrace{r^N\min\limits_{\partial\Delta(\alpha,r)}|g(z)|}_{r'})$: $f(z)-w=f(\alpha)+(z-\alpha)^N g(z)-w=$

$=\underbrace{(z-\alpha)^N g(z)}_{\text{"}f\text{"}}+\underbrace{(f(\alpha)-w)}_{\text{"}g\text{"}}$

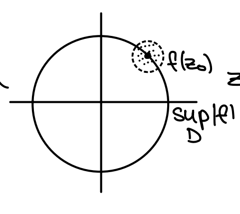
Rouchéjev izrek:

$f(z)-w$ ima toliko ničel na $\Delta(\alpha,r)$, kolikor jih ima $(z-\alpha)^N g(z)$.

Torej ima na $\Delta(\alpha,r)$ ničlo, oz. $w\in f(\Delta(\alpha,r))$. $\square$

OPOMBA: (dokaz principa maksima)

D območje, $f:D\to\mathbb{C}$ omejena holomorfna funkcija: $|f(z)|<\sup\limits_{D}|f|$, $\forall z\in D$ ali f je konstantna.



$z_0\in D$.

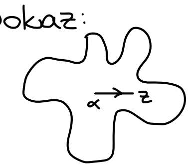
slika f je v sliki $f(D)$

f odprta, če ni konstantna.

TRDITEV (obstoj logaritma):

Naj bo D zvezdasto domočje in $f \in \mathcal{H}(D)$ brez ničel.
Tedaj obstaja $g \in \mathcal{H}(D)$, da je $f = e^g$.
($g = \log f$; $e^g = e^{g_1}$
$e^{g-g_1} = 1$; $g_1 = g + 2k\pi i$; $k \in \mathbb{Z}$)

Dokaz:



$\alpha \in D$ iz katere "vidimo" vse ostale točke

Definiramo: $g(z) = \int_{[\alpha, z]} \frac{f'(\xi)}{f(\xi)} \, d\xi$.

Kot v dokazu Morerovega izreka sledi, da je $g \in \mathcal{H}(D)$ in $g'(z) = \frac{f'(z)}{f(z)}$.
Oglejmo si: $(f \cdot e^{-g})' = f' e^{-g} + f e^{-g}(-g') = e^{-g}\left(f' - f \cdot \frac{f'}{f}\right) = 0$
$\Rightarrow$ fe^{-g} je konstantna funkcija.
$fe^{-g} = A \neq 0 \Rightarrow f = e^{g+B}$ $\square$

OPOMBA: $f^\alpha = e^{\alpha g}$

HOLOMORFNE FUNKCIJE KOT PRESLIKAVE

$f: D \xrightarrow{\text{območje}} \mathbb{C}$; f ni konstantna, $f \in \mathcal{H}(D)$ $\Rightarrow$ f je odprta preslikava

IZREK: Naj bo $D \subseteq \mathbb{C}$ odprta in $f \in \mathcal{H}(D)$ ter $\alpha \in D$ taka točka, da je $f'(\alpha) \neq 0$.
Tedaj obstaja okolica $U \subseteq D$ točke α in okolica V točke $f(\alpha)$, da je
$f: U \to V$ BIHOLOMORFIZEM (f bijekcija, f^{-1} holomorfna)

Dokaz:
$d_\alpha f = f'(\alpha) \, d_\alpha z$
$f'(\alpha) \neq 0$ je to nesingularna
$\mathbb{R}$ (in tudi $\mathbb{C}$) linearna preslikava.
Uporabimo izrek o inverzni preslikavi za $D \subseteq \mathbb{R}^2$:
$\exists U$ okolica α in $\exists V$ okolica $f(\alpha)$: $f: U \to V$ difeomorfizem
$f^{-1} = g: V_w \to U_z$ gladek inverz.

(1) Vemo $f \circ g = id_V$.
Odvajamo po $\overline{w}$: $f_z g_{\overline{w}} + f_{\overline{z}} (\overline{g})_w = 0$
$f_{\overline{z}}$: 0, ker $f \in \mathcal{H}(D)$
$f_z = f'$, $f_z(\alpha) \neq 0$, $f_z(z) \neq 0$ na U $\Rightarrow g_{\overline{w}} = 0$
$\Rightarrow g$ holomorfna na V $\square$

(2) $\frac{g(w) - g(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} \xrightarrow{z \to z_0} \frac{1}{f'(z_0)}$; $f(z) = w$, $f(z_0) = w_0$ $\square$

IZREK (izrek o lokalni strukturi holomorfnih funkcij):
Naj bo D območje, $f \in \mathcal{H}(D)$ nekonstantna in $\alpha \in D$. Naj bo N stopnja ničle funkcije $z \mapsto f(z) - f(\alpha)$ v točki α.
Potem obstaja taka okolica U točke α v D; $\Phi \in \mathcal{H}(U)$ in $r > 0$, da je:
(1) $f(z) = f(\alpha) + \Phi(z)^N$ na U
(2) $\Phi'(z) \neq 0$ na U, $\Phi(\alpha) = 0$ in $\Phi: U \to \Delta(0, r)$ biholomorfizem.

Komentar: $f(z)=f(\alpha)+\Phi(z)^N$

$$z \overset{\text{biholomorfizem}}{\mapsto} \underset{w}{\Phi(z)} \longmapsto \underset{w^N}{\Phi(z)^N} \longrightarrow \underset{f(\alpha)+w^N}{f(\alpha)+\Phi(z)^N}$$

Lokalno gledano je f ekvivalentna preslikavi $w \mapsto w^N$ (če ni konstantna)

Dokaz:

Ker f ni konstantna in D območje, obstaja $N \in \mathbb{N}$, da je $f(z)=f(\alpha)+(z-\alpha)^N g(z)$; $g(z) \neq 0$ na $\overline{\Delta(\alpha,R)} \subseteq D$.

$\exists h \in \mathcal{H}(\Delta(\alpha,R)):\ h^N=g$

$\Phi(z)=(z-\alpha)h(z)$

$\Phi'(z)=h(z)+(z-\alpha)h'(z) \Rightarrow \Phi'(\alpha)=h(\alpha)\neq 0$

Po izreku o inverzni preslikavi, obstaja U okolica točke α in $r>0$, da je:

$\Phi: U \to \Delta(\alpha,r)$ biholomorfna

$\Phi' \neq 0$ na U

$\Phi(\alpha)=0$. $\square$

POSLEDICA: Naj bo $f: D \to f(D)$ injektivna holomorfna preslikava.

Tedaj je: (1) $f' \neq 0$ na D;

(2) f je biholomorfizem.

Dokaz:

f injektivna $\Rightarrow$ f nekonstantna na vsaki komponenti D za povezanost

(f je odprta preslikava in $f(D)$ odprta podmnožica $\mathbb{C}$)

$\alpha \in D$:

lokalno: $f(z)=f(\alpha)+\Phi(z)^N$; kjer $\Phi: \alpha \in U \mapsto \Delta(a,r)$ biholomorfna

lokalno se f obnaša kot $w \mapsto w^N$; injektivna $\Leftrightarrow N=1 \Leftrightarrow$ odvod $\neq 0$

$\Rightarrow f'(\alpha)\neq 0 \Rightarrow f' \neq 0$

$f: D \to f(D)$ bijekcija $\Rightarrow$ obstaja inverz $f^{-1}: f(D) \to D$, lokalno holomorfen (po izreku o inverzni preslikavi za holomorfne funkcije

$\Rightarrow f$ biholomorfizem $\square$

OPOMBA: $z \mapsto e^z = f(z)$; $f'(z)=e^z \neq 0$; f ni injektivna

MÖBIUSOVE TRANSFORMACIJE oz. LOMLJENE LINEARNE PRESLIKAVE

$a,b,c,d \in \mathbb{C}$: $f: \mathbb{CP}^1 \to \mathbb{CP}^1$, $z \mapsto \frac{az+b}{cz+d}$

pol: $-\frac{d}{c}$ $(c\neq 0)$ $\to f(-\frac{d}{c})=\infty$

$c=0, d\neq 0 \to f$ ima pol v $\infty \to f(\infty)=\infty$

$f(\infty)=\frac{a}{c}$ $(c\neq 0)$

I) $a=b=c=d=0$

$\to$ Hočemo, da je vsaj eno od teh števil različno od 0:

II) Denimo $d\neq 0$:

i) $ad-cb=0 \to a=\frac{cb}{d} \to \frac{\frac{cb}{d}z+b}{cz+d}=\frac{\frac{b}{d}(cz+d)}{cz+d}=\frac{b}{d}$

ii) $ad-cb\neq 0$

Predpostavimo: $ad-cb=1$

$a,b,c,d\in\mathbb{C} \wedge ad-bc\neq 0$ (oz. $ad-bc=1$)

$$z \xmapsto{f} \frac{az+b}{cz+d} = w \qquad \rightarrow \qquad \begin{bmatrix} a & b\\ c & d\end{bmatrix} \qquad ad-bc=1$$

$$az+b = w(cz+d)$$

$$(a-cw)z = wd-b \rightarrow z = \frac{dw-b}{-cw+a} \rightarrow \begin{bmatrix} d & -b\\ -c & d\end{bmatrix} = \begin{bmatrix} a & b\\ c & d\end{bmatrix}^{-1}$$

$$w \xmapsto{g} \frac{\alpha w+\beta}{\gamma w+\delta} = \frac{\alpha\frac{az+b}{cz+d}+\beta}{\gamma\frac{az+b}{cz+d}+\delta} = \frac{(\alpha a+\beta c)z+(\alpha b+\beta d)}{(\gamma a+\delta c)z+(\gamma b+\delta d)} \rightarrow \begin{bmatrix} \alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix}$$

$\mathrm{Aut}(\mathbb{CP}^1)=\left\{z\mapsto \frac{az+b}{cz+d} \;\middle|\; a,b,c,d\in\mathbb{C},\ ad-cb=1\right\}$ je grupa za kompozitum.

OSNOVNE MÖBIUSOVE TRANSFORMACIJE:

(1) translacija: $z\mapsto z+b \rightarrow \begin{bmatrix} 1 & b\\ 0 & 1\end{bmatrix}$

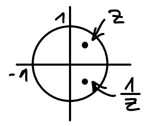
(2) množenje z neničelnim številom: $a\neq 0$

$z\mapsto az \rightarrow \begin{bmatrix} a & 0\\ 0 & 1\end{bmatrix}$ oz. $z\mapsto \frac{\alpha}{\frac{1}{\alpha}}\cdot z \rightarrow \begin{bmatrix} \alpha & 0\\ 0 & \frac{1}{\alpha}\end{bmatrix}$

i) $a>0$: razteg

ii) $|a|=1 \wedge a=e^{i\varphi}$: $z\mapsto e^{i\varphi}z$ ← rotacija

(3) inverzija na krožnico: $z\mapsto \frac{1}{\bar z} = \frac{z}{|z|^2}$



TRDITEV: Möbiusove transformacije tipa (1), (2), (3) generirajo $\mathrm{Aut}(\mathbb{CP}^1)$.

Dokaz:

$z\mapsto \frac{az+b}{cz+d}$

$$c\neq 0:\ z \xmapsto{(2)} cz \xmapsto{(1)} cz+d \xmapsto{(3)} \frac{1}{cz+d} \xmapsto{(2)} \frac{b-\frac{ad}{c}}{cz+d} \xmapsto{(1)} \frac{b-\frac{ad}{c}}{cz+d}+\frac{a}{c} = \frac{bc-da+a(cz+d)}{c(cz+d)} = \frac{az+b}{cz+d}$$

$$c=0:\ z \xmapsto{(2)} \frac{a}{d}z \xmapsto{(1)} \frac{a}{d}z+\frac{b}{d} = \frac{az+b}{d}$$

□

TRDITEV: Naj bodo $\alpha_1,\beta_1,\gamma_1$ trije različni elementi Riemannove sfere in $\alpha_2,\beta_2,\gamma_2$ tudi trije različni elementi $\mathbb{CP}^1$.
Tedaj obstaja Möbiusova transformacija f, ki slika: $\alpha_1\mapsto\alpha_2$, $\beta_1\mapsto\beta_2$ in $\gamma_1\mapsto\gamma_2$.

Dokaz:

$\alpha,\beta,\gamma\in\mathbb{CP}^1$ paroma različni, $0,1,\infty$

$$\begin{tikzcd} \alpha_1 \arrow[r, mapsto, "f_1"'] & 0 & \alpha_2 \arrow[l, mapsto, "f_2"] \\ \beta_1 \arrow[r, mapsto] & 1 & \beta_2 \arrow[l, mapsto] \\ \gamma_1 \arrow[r, mapsto] \arrow[rr, bend right, "f_2^{-1}\circ f_1"'] & \infty & \gamma_2 \arrow[l, mapsto] \end{tikzcd}$$

$$\left(\frac{\beta-\gamma}{\beta-\alpha}\right)\frac{z-\alpha}{z-\gamma}$$

$\alpha\mapsto 0$

$\beta\mapsto 1$

$\gamma\mapsto\infty$

… $\alpha=\infty$, $\beta,\gamma\in\mathbb{C}$: $\frac{\beta-\gamma}{z-\gamma}$: $\infty\mapsto 0$, $\beta\mapsto 1$, $\gamma\mapsto\infty$

… $\beta=\infty$, $\alpha,\gamma\in\mathbb{C}$: $\frac{z-\alpha}{z-\gamma}$: $\alpha\mapsto 0$, $\infty\mapsto 1$, $\gamma\mapsto\infty$

… $\gamma=\infty$, $\alpha,\beta\in\mathbb{C}$: $\frac{z-\alpha}{\beta-\alpha}$: $\alpha\mapsto 0$, $\beta\mapsto 1$, $\infty\mapsto\infty$

□

TRDITEV: Vsaka Möbiusova transformacija slika množico premic in krožnic iz $\mathbb{C}$ v množico premic in krožnic iz $\mathbb{C}$.
$\{$premice, krožnice v $\mathbb{C}\}$ $\circlearrowleft f$ Möbiusova

(premica v $\mathbb{C}$ $\sim$ krožnica na Riemannovi sferi skozi točko ∞)

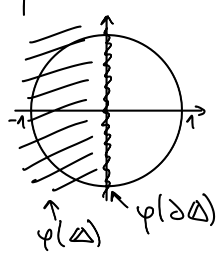
Naj bo $f: \mathbb{CP}^1 \to \mathbb{CP}^1$ Möbiusova transformacija.
K krožnica v $\mathbb{C} \xmapsto{f}$ premica v $\mathbb{C}$; če ima f na K pol.
p premica v $\mathbb{C} \xmapsto{f}$ premica v $\mathbb{C}$; če ima f na p pol (v $\mathbb{C}$ ali ∞)
p premica v $\mathbb{C} \xmapsto{f}$ krožnica v $\mathbb{C}$; če f na p nima pola (ne v $\mathbb{C}$ ne v ∞)

Dokaz:
Vsaka Möbiusova transformacija je kompozitum osnovnih treh transformacij:
(1) translacija: premice $\to$ premice, krožnice $\to$ krožnice
(2.i) raztegi: premice $\to$ premice, krožnice $\to$ krožnice
(2.ii) rotacije: premice $\to$ premice; krožnice $\to$ krožnice
(3) inverzija na krožnico $(z \to \frac{1}{z})$
$(ax+by=c;\ (a,b)\neq(0,0))$
$\underbrace{\mathrm{Re}(\alpha z)=c}_{\text{enačba premice v } \mathbb{C}}$; $\alpha = a-ib$; $\alpha \neq 0$ normala na premico
$z = x+iy$

$z \mapsto \frac{1}{z} = w \to z = \frac{1}{w}$
premica: $\mathrm{Re}(\alpha \frac{1}{w}) = c$
$\bar{\alpha}\frac{1}{\bar{w}} + \alpha\frac{1}{w} = 2c$
$\bar{\alpha}w + \alpha\bar{w} = 2cw\bar{w} = 2c|w|^2$
i) $c=0$: $\mathrm{Re}(\alpha w)=0$ $\leftarrow$ začetna premica gre skozi 0 (pol $z \to \frac{1}{z}$)
ii) $c\neq 0$: $w = u+iv$; $u,v \in \mathbb{R}$
$(a-ib)(u-iv)+(a+ib)(u+iv) = 2(u^2+v^2)c$
$au-bv = c(u^2+v^2)$
$u^2+v^2-\frac{a}{c}u+\frac{b}{c}v=0$ $\leftarrow$ krožnica s središčem v $(\frac{a}{2c}, -\frac{b}{2c})$ skozi 0.
$(u-\frac{a}{2c})^2+(v+\frac{b}{2c})^2 = \frac{a^2+b^2}{4c^2}$

krožnica: $|z-\alpha| = r$
$(z-\alpha)(\bar{z}-\bar{\alpha}) = r^2$
$|z|^2 - \alpha\bar{z} - \bar{\alpha}z + (|\alpha|^2-r^2) = 0 \qquad (z = \frac{1}{w})$
$\frac{1}{|w|^2} - \alpha\frac{1}{\bar{w}} - \bar{\alpha}\frac{1}{w} + (|\alpha|^2-r^2) = 0$
$1 - \alpha w - \bar{\alpha}\bar{w} + (|\alpha|^2-r^2)|w|^2 = 0$
i) $|\alpha|^2 = r^2$ (krožnica skozi točko 0)
$\to 2\mathrm{Re}(\alpha w) = 1$ $\leftarrow$ premica
ii) $|\alpha|^2 \neq r^2 \to |w|^2 - \frac{\alpha}{|\alpha|^2-r^2}w - \frac{\bar{\alpha}}{|\alpha|^2-r^2}\bar{w} + \frac{1}{|\alpha|^2-r^2} = 0$
krožnica s središčem v $(\frac{\bar{\alpha}}{|\alpha|^2-r^2}, \frac{r}{||\alpha|^2-r^2|})$ $\square$

Zgled:
$\varphi(z) = \frac{z+1}{z-1}$. Zanima nas: $\varphi(\partial\Delta)$ in $\varphi(\Delta)$.



$\varphi(\partial\Delta)$ = premica (imaginarna os)
$\varphi(-1) = 0$
$\varphi(i) = \frac{i+1}{i-1} = \frac{(i+1)(-i-1)}{2} = \frac{1-i-i-1}{2} = -i$

$\varphi(\Delta)$ = leva polravnina

KONFORMNE PRESLIKAVE

Izometrija: $F:(M,d)\to(N,\rho)$ metrična prostora.

Izometrija ohranja metrike: $d(x,y)=\rho(F(x),F(y))$, $\forall x,y\in M$

$(\mathbb{R}^2,d_2)\circlearrowleft F$ izometrija ravnine

$d_2(x,y)=d_2(F(x),F(y))$, $\forall x,y\in\mathbb{R}^2$

$\|x-y\|_2=\|F(x)-F(y)\|_2$

Denimo: $F(0)=0$ (kompozitum s translacijo)

$\|x\|_2=\|F(x)\|_2$ $\forall x$ ohranja normo

$(x-y,\,x-y)=(F(x)-F(y),\,F(x)-F(y))$

$\|x\|^2-2(x,y)+\|y\|^2=\|F(x)\|^2-2(F(x),F(y))+\|F(y)\|^2$

$\Rightarrow (x,y)=(F(x),F(y))$ $\forall x,y$

$\Rightarrow$ F ohranja kote

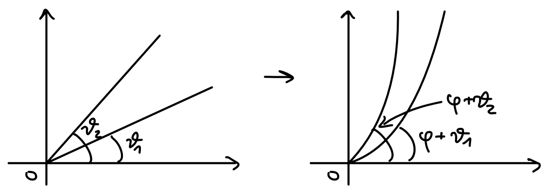
DEFINICIJA: Naj bo $f: D^{odp}\subseteq\mathbb{C}\to\mathbb{C}$ funkcija in $\alpha\in D$.
Rečemo, da f OHRANJA KOTE v α, če obstaja tak $\varphi\in[0,2\pi)$, da za vsak $\vartheta\in[0,2\pi)$ velja:

$$\lim_{r\searrow 0}\frac{f(\alpha+re^{i\vartheta})-f(\alpha)}{|f(\alpha+re^{i\vartheta})-f(\alpha)|}=e^{i\vartheta}e^{i\varphi}.$$

Če f ohranja kote v vsaki točki iz D, je KONFORMNA na D.

OPOMBA: $\alpha=0$, $f(\alpha)=0$: $\lim\limits_{r\searrow 0}\frac{f(re^{i\vartheta})}{|f(re^{i\vartheta})|}=e^{i\varphi}e^{i\vartheta}$

r majhen: $f(re^{i\vartheta})=\underbrace{|f(re^{i\vartheta})|}_{\text{razteg}}\,\underbrace{e^{i\varphi}}_{\text{rotacija za }\varphi}e^{i\vartheta}$



Konformnost ohranja kote in orientacijo.

Zgled:

1) $z\mapsto\bar z$ ni konformna

$\alpha=0$: $\overset{?}{\exists}\varphi$: $\lim\limits_{r\searrow 0}\frac{f(re^{i\vartheta})}{|f(re^{i\vartheta})|}=e^{i\varphi}e^{i\vartheta}$ $\forall\vartheta$?

$\lim\limits_{r\searrow 0}\frac{re^{i\vartheta}}{r}=e^{i\vartheta}$

2) $z\mapsto z^2$

$\lim\limits_{r\searrow 0}\frac{r^2e^{i2\vartheta}}{r^2}=e^{2i\vartheta}\overset{?}{=}e^{i\varphi}e^{i\vartheta}$

IZREK: Naj bo $f:D\to\mathbb{C}$ funkcija.

1) Če je $f\in\mathcal{H}(D)$ in $f'(z)\neq 0$, $\forall z\in D$, je f konformna na D.
2) Naj bo f diferenciabilna in konformna na D.
Tedaj je $f\in\mathcal{H}(D)$ in $f'(z)\neq 0$, $\forall z\in D$.

Dokaz:

(1): $\alpha\in D$: $\lim\limits_{r\to 0}\frac{f(\alpha+re^{i\vartheta})-f(\alpha)}{|f(\alpha+re^{i\vartheta})-f(\alpha)|} \overset{f\in\mathcal{H}(D)}{=} \lim\limits_{r\to 0}\frac{f'(\alpha)re^{i\vartheta}+o(r)}{|f'(\alpha)re^{i\vartheta}+o(r)|} \overset{f'(\alpha)\neq 0 \Rightarrow \lim \frac{o(r)}{r}=0}{=} \frac{f'(\alpha)}{|f'(\alpha)|}e^{i\vartheta}=e^{i\varphi}e^{i\vartheta}$; $\varphi=\arg f'(\alpha)$

(2): $f(\alpha+re^{i\vartheta})-f(\alpha) = \underbrace{f_z(\alpha)re^{i\vartheta}+f_{\bar z}(\alpha)re^{-i\vartheta}}_{d_\alpha f(re^{i\vartheta})}+o(r)$ (f diferenciabilna)

i) $d_\alpha f = 0 \Rightarrow f_{\bar z}(\alpha)=0$

ii) $d_\alpha f \neq 0$

obstajata največ dva kota ϑ_0, $\pi+\vartheta_0$, da je $(d_\alpha f)(e^{i\vartheta_0})=0$ in $(d_\alpha f)(e^{i(\pi+\vartheta_0)})=0$

$\vartheta \neq \vartheta_0, \vartheta_0+\pi$

f je konformna: $\lim\limits_{r\to 0}\frac{f(\alpha+re^{i\vartheta})-f(\alpha)}{|f(\alpha+re^{i\vartheta})-f(\alpha)|}=e^{i\varphi}e^{i\vartheta}$ za nek $\varphi\in[0,2\pi)$

$$\frac{f_z(\alpha)e^{i\vartheta}+f_{\bar z}(\alpha)e^{-i\vartheta}}{|f_z(\alpha)e^{i\vartheta}+f_{\bar z}(\alpha)e^{-i\vartheta}|} = e^{i\varphi}e^{i\vartheta} \quad \forall\vartheta\notin\{\vartheta_0, \vartheta_0+\pi\}$$

$f_{\bar z}(\alpha)^2e^{-2i\vartheta}$

$$(f_z(\alpha)e^{i\vartheta}+f_{\bar z}(\alpha)e^{-i\vartheta})^2 = e^{2i\varphi}e^{2i\vartheta}\left(f_z(\alpha)e^{i\vartheta}+f_{\bar z}(\alpha)e^{-i\vartheta}\right)\left(\overline{f_z(\alpha)}e^{-i\vartheta}+\overline{f_{\bar z}(\alpha)}e^{i\vartheta}\right)$$

(desna stran nima člena, ki bi vseboval $e^{-2i\vartheta}$

$\Rightarrow f_{\bar z}(\alpha)=0 \rightarrow f_{\bar z}=0$ na D

$\Rightarrow f\in\mathcal{H}(D)$

Če je odvod f v neki točki $\alpha\in D$ enak 0: $f'(\alpha)=0$, potem je lahko f ekvivalentna preslikavi $w\mapsto w^N$, $N\geq 2$, ki ne ohranja kotov v 0.

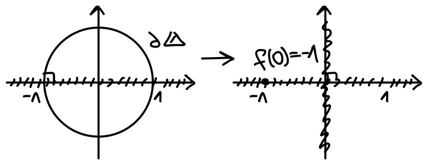
$\Rightarrow f'(z)\neq 0 \ \forall z$ $\square$

Zgled:

$f(z)=\frac{z+1}{z-1}$

$f(\Delta)$, $f(\partial\Delta)$ $\qquad f(-1)=0$

$f(\mathbb{R}\setminus\{1\})\subseteq\mathbb{R}$ $\qquad f(i)$

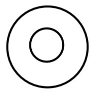
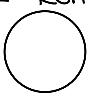


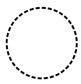
$f: D\to\Omega=f(D)$ bijektivna in holomorfna ⇒ biholomorfna

⇒ $f'(z)\neq 0$ $\forall z$ ⇒ tak f je konformna preslikava.

DEFINICIJA: Odprti množici D in Ω sta KONFORMNO EKVIVALENTNI, če obstaja $f: D\to\Omega$ biholomorfna preslikava.

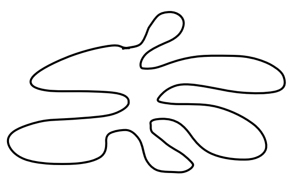
OPOMBE:
1) To je ekvivalenčna relacija na odprtih podmnožicah $\mathbb{C}$.
2) Če sta D in Ω konformno ekvivalentni, sta homeomorfni.

 in  nista konformno ekvivalentni

3)  ← Δ in $\mathbb{C}\ (=\mathbb{R}^2)$ sta homeomorfna

$f:\mathbb{C}\to\Delta$ holomorfna; $|f(z)|<1$ omejena ⇒ konstantna

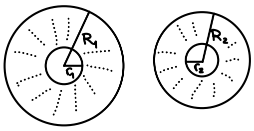
OPOMBA: D je enostavno povezano območje, če nima "lukenj".
(npr. vsako zvezdasto/konveksno območje)



$\left.\begin{array}{l}\Delta\\ \mathbb{C}\\ \mathbb{CP}^1\end{array}\right\}$ enostavno povezana območja } topološko enaka

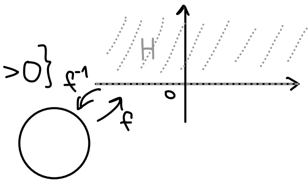
$\mathbb{CP}^1$: kompakten prostor; topološko ni enak ostalima dvema

OPOMBA: $A(R_1,r_1)$, $A(R_2,r_2)$ konformno ekvivalentna $\Leftrightarrow \frac{R_1}{r_1}=\frac{R_2}{r_2}$



$z\mapsto \frac{r_2}{r_1}z$

Pogosto je modelno območje zgornja polravnina: $H=\{z\in\mathbb{C}\mid \operatorname{Im} z>0\}$



$z\overset{f}{\mapsto} -i\frac{z+1}{z-1}=w$ in inverz $w\overset{f^{-1}}{\mapsto}\frac{iw+1}{iw-1}$

$(i\mapsto 0)$

IZREK (Riemannov upodobitveni izrek)

Naj bo $D\subsetneq\mathbb{C}$ enostavno povezano območje, ki ni $\mathbb{C}$.

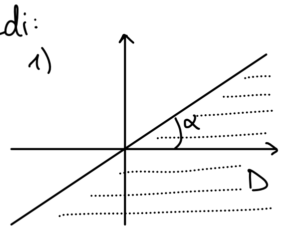
Tedaj je D konformno ekvivalentna Δ.

$(\exists f: D\to\Delta$ biholomorfna; $f^{-1}:\Delta\to D)$

Dokaz: (*)

Zgledi:

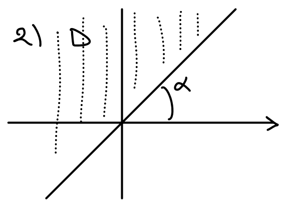
1)



$D\to H$

$z\mapsto -e^{-i\alpha}z$ ← rotacija za $(\pi-\alpha)$

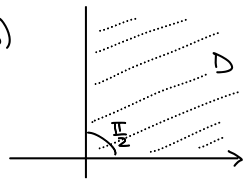
2)



$D\to H$

$z\mapsto e^{-i\alpha}z$ ← rotacija za α

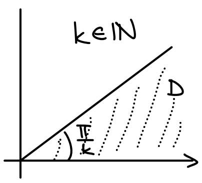
3)



$D\to H$

$z\mapsto z^2$

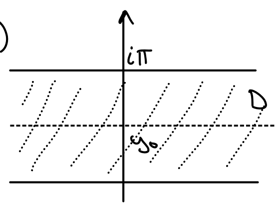
4)



$D\to H$

$z\mapsto z^k$

5)



$D\to H$

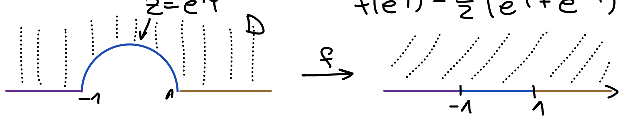
$z\mapsto e^z=e^xe^{iy}$; $z=x+iy$

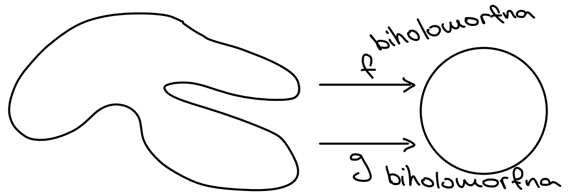
6) $f: D\to H$

$z\mapsto \frac{1}{2}\left(z+\frac{1}{z}\right)$

$f'(z)=\frac{1}{2}\left(1-\frac{1}{z^2}\right)$

$f(e^{i\varphi})=\frac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)=\operatorname{Re}(e^{i\varphi})=\cos\varphi$





$f\circ g^{-1}:\Delta\to\Delta$ biholomorfna

Iščemo vse holomorfne avtomorfizme Δ: $\mathrm{Aut}(\Delta)=\{f:\Delta\to\Delta \mid \text{bijektivna, holomorfna}\}$

IZREK: $\mathrm{Aut}(\mathbb{C})=\{\alpha z+\beta \mid \alpha\in\mathbb{C}\setminus\{0\},\ \beta\in\mathbb{C}\}$ (Aut(D) grupa za kompozitum) (4 realni parametri)

Dokaz:
$f:\mathbb{C}\to\mathbb{C}$ biholomorfna
- Če $f(z)=\alpha z+\beta$, $\alpha\neq 0$, $f^{-1}(w)=\frac{w-\beta}{\alpha}$; potem je $\mathrm{Aut}(\mathbb{C})\supseteq\{f(z)=\alpha z+\beta,\ \alpha\neq 0,\ \alpha,\beta\in\mathbb{C}\}$.
- Naj bo $f\in\mathrm{Aut}(\mathbb{C})$.

Točka ∞ je izolirana singularnost za f:
1) bistvena singularnost:
   i) slika vsake prebodene okolice ∞ je gosta v $\mathbb{C}$
   ii) f ni konstantna $\Rightarrow$ f odprta preslikava
   $\underbrace{f(\Delta)}_{\text{odprta}}\cap \underbrace{f(\mathbb{C}\setminus\overline{\Delta})}_{\text{gosta v } \mathbb{C}}\neq 0$ $\Rightarrow$ ∞ ni bistvena singularnost
2) odpravljiva singularnost ne more biti, ker bi bila f omejena in zato konstantna (Liouville)
3) f ima v ∞ pol $\Rightarrow$ f polinom, $f'(z)\neq 0$, $\forall z$ $\to$ f' konstantna
   $\Rightarrow f(z)=\alpha z+\beta$, $\alpha\neq 0$ $\square$

TRDITEV: $\mathrm{Aut}(\mathbb{C}\setminus\{0\})=\mathrm{Aut}(\mathbb{C}^*)=\{z\mapsto\alpha z,\ \alpha\neq 0\}\cup\{z\mapsto\frac{\alpha}{z}\}$

Dokaz:
$f:\mathbb{C}^*\to\mathbb{C}^*$: $0,\infty$ izolirani singularnosti:
- nobena ni bistvena singularnost
- obe ne moreta biti odpravljivi
- ne moreta biti obe pola
- ∞ pol, 0 odpravljiva: $f(0)=0\Rightarrow z\mapsto\alpha z$, $\alpha\neq 0$
- ∞ ničla, 0 pol: $z\mapsto\frac{\alpha}{z}$ $\square$

TRDITEV: $\mathrm{Aut}(\mathbb{C}P^1)=\{z\mapsto\frac{az+b}{cz+d};\ ad-bc=1\}$ (6 realnih parametrov)
Riemannova sfera

Dokaz:
$(\supseteq)$: očitno
$(\subseteq)$: $f:\mathbb{C}P^1\to\mathbb{C}P^1$ avtomorfizem
$$\left.\begin{array}{l}0\mapsto\alpha\\ 1\mapsto\beta\\ \infty\mapsto\gamma\end{array}\right\}\in\mathbb{C}P^1,\text{ paroma različni}$$
Obstaja Möbiusova transformacija φ, ki slika: $\alpha\mapsto 0$, $\beta\mapsto 1$, $\gamma\mapsto\infty$.
Poglejmo: $g=\varphi\circ f\in\mathrm{Aut}(\mathbb{C}P^1)$: $0\mapsto 0$, $1\mapsto 1$, $\infty\mapsto\infty$ $\Rightarrow g\in\mathrm{Aut}(\mathbb{C})$ $\Rightarrow g=id$ $\Rightarrow f=\varphi^{-1}$ $\square$

Kandidati za $\mathrm{Aut}(\Delta)$:
1) $z\mapsto\frac{\alpha}{\bar z}$; $|\alpha|=1$, $\alpha=e^{i\vartheta}$
   $z\mapsto e^{i\vartheta}z$; $\vartheta\in[0,2\pi)$
2) $a\in\Delta$, $|a|<1$: $z\mapsto\frac{a-z}{1-\bar a z}=\psi_a(z)$ $\left(=\psi_a^{-1}(z)\right)$: $0\mapsto a$; $\infty\mapsto\frac{a}{\bar a}$
   $|z|=1$: $\frac{a-z}{1-\bar a z}\cdot\frac{\bar a-\bar z}{1-a\bar z}=\frac{|a|^2-\bar a z-a\bar z+1}{1-a\bar z-\bar a z+|a|^2}=1=|\psi_a(z)|^2$

IZREK (Schwarzova lema):

Naj bo $f \in \mathcal{H}(\Delta)$ taka, da velja: 1) $f: \Delta \to \overline{\Delta}$ in

2) $f: 0 \mapsto 0$.

Potem velja: (1) $|f(z)| \leq |z|$, $\forall z \in \Delta$ in

(2) $|f'(0)| \leq 1$.

DODATEK: Če v (1) velja enakost za nek $z \in \Delta \setminus \{0\}$, ali če velja enakost v (2), obstaja tak $\vartheta \in [0, 2\pi)$, da je $f(z) = e^{i\vartheta} z$.

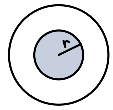
Dokaz:

Naj bo $g(z) = \frac{f(z)}{z} \in \mathcal{H}(\Delta \setminus \{0\})$.

g ima v 0 izolirano singularnost

$\to$ 0 odpravljiva singularnost: $\lim\limits_{z\to 0} g(z) = \lim\limits_{z\to 0} \frac{f(z)}{z} = \lim\limits_{z \to 0} \frac{f(z)-f(0)}{z-0} = f'(0)$

$$g(z) = \begin{cases} \frac{f(z)}{z} ; & z \in \Delta \setminus \{0\} \\ f'(0) ; & z = 0 \end{cases} \quad \in \mathcal{H}(\Delta)$$



$0 < r < 1$: $\Delta(0, r)$

princip maksima: za $|z| \leq r$ je: $|g(z)| \leq \max\limits_{|z|=r} |g(z)| \leq \frac{1}{r}$; $\forall r \in (0,1)$ $\Rightarrow |g(z)| \leq 1$ na Δ

$\Rightarrow \frac{f(z)}{z} \leq 1 = |f(z)| \leq |z|$; $\forall z$ in $|f'(0)| \leq 1$.

Če v (1) velja enakost za nek $z \in \Delta \setminus \{0\}$ ali enakost v (2).

Po principu maksima za g: $g =$ konstantna na Δ.

$\frac{f(z)}{z} = g = e^{i\vartheta}$; $\vartheta \in [0, 2\pi)$ $\square$

IZREK: $\mathrm{Aut}(\Delta) = \{ z \mapsto e^{i\vartheta} \frac{a-z}{1-\bar{a}z} ; \vartheta \in [0, 2\pi), |a| < 1 \}$ (3 realni parametri)

Dokaz:

Naj bo $f \in \mathrm{Aut}(\Delta)$ in $a = f(0) \in \Delta$.

Poglejmo: $F(z) = (\psi_a \circ f)(z) \in \mathrm{Aut}(\Delta)$

$F(0) = 0$

(Schwarzova lema) $\Rightarrow$ $|F'(0)| \leq 1$

$F^{-1} \in \mathrm{Aut}(\Delta)$; $F^{-1}(0) = c \to |F^{-1}{}'(0)| \leq 1$

$\left|\frac{1}{F'(0)}\right| \leq 1 \Rightarrow 1 \leq |F'(0)|$

$\Rightarrow |F'(0)| = 1$

(dodatek) $\Rightarrow$ $F(z) = e^{i\vartheta} z$; $\vartheta \in [0, 2\pi)$ $(= (\psi_a \circ f)(z))$

$f(z) = \psi_a^{-1}(e^{i\vartheta} z) = \frac{a - e^{i\vartheta} z}{1 - \bar{a} e^{i\vartheta} z} = e^{i\vartheta} \frac{a e^{-i\vartheta} - z}{1 - \overline{a e^{-i\vartheta}} z} = \psi_{a e^{-i\vartheta}}(z)$ $\square$

$D \xrightarrow{f} \Delta$

$D \xrightarrow{g} \Delta$ $\to f \circ g^{-1} \in \mathrm{Aut}(\Delta)$ $\quad e^{i\vartheta} \frac{a-z}{1-\bar{a}z}$

TRDITEV: Naj bo D enostavno povezano območje v $\mathbb{C}$, ki ni enako $\mathbb{C}$, in $\alpha \in D$. Tedaj obstaja natanko en biholomorfizem $F: D \to \Delta$: $\alpha \mapsto 0$; $F'(\alpha) > 0$.

Dokaz:

ENOLIČNOST: $F_1, F_2 : D \to \Delta$

$\alpha \mapsto 0$

$F_1'(\alpha) > 0$, $F_2'(\alpha) > 0$

$G = F_1 \circ F_2^{-1} \in \mathrm{Aut}(\Delta)$: $0 \mapsto 0$ $\Rightarrow G(z) = e^{i\vartheta} z$; $\vartheta \in [0, 2\pi)$

$(F_1 \circ F_2^{-1})(z) = e^{i\vartheta} z$

$F_1(z) = e^{i\vartheta} F_2(z) \to \underset{\substack{\vee \\ 0}}{F_1'(\alpha)} = e^{i\vartheta} \underset{\substack{\vee \\ 0}}{F_2'(\alpha)} \Rightarrow \underset{\substack{\shortparallel \\ 1}}{e^{i\vartheta}} > 0 \Rightarrow \vartheta = 0$

$e^{i\vartheta} = \frac{F_1'(\alpha)}{F_2'(\alpha)} > 0 \Rightarrow F_1 = F_2$

OBSTOJ: Naj bo $H: D \to \Delta$ biholomorfna (obstaja po Riemannovem izreku)

$$\alpha \mapsto a \in \Delta$$

$$\psi_a(z) = \frac{a-z}{1-\bar{a}z} \qquad \psi_a(0)$$

$$\psi_a \circ H = \psi_a^{-1} \circ H : D \to \Delta, \quad \alpha \mapsto 0$$

$$H_1 = \psi_a \circ H : D \to \Delta \text{ biholomorfna}, \quad \alpha \mapsto 0$$

$$H_1'(\alpha) \neq 0 \quad \text{in} \quad H_1'(\alpha) = e^{i\vartheta}|H_1'(\alpha)|$$

$$F(z) = e^{-i\vartheta} H_1(z)$$

$$F: D \to \Delta \text{ biholomorfna}, \quad \alpha \mapsto 0$$

$$F'(\alpha) = e^{-i\vartheta} H_1'(\alpha) = |H_1'(\alpha)| > 0 \qquad \square$$

(✱) Dokaz (Riemannov upodobitveni izrek)

$F: D \to \Delta$ biholomorfna, $\alpha \mapsto 0$ in $f: D \to \Delta$ holomorfna, $\alpha \mapsto 0$

$g = f \circ F^{-1}: \Delta \to \Delta$ holomorfna, $0 \mapsto 0$

(Schwarzova lema): $|g'(0)| \leq 1$

$$g'(0) = f'(F^{-1}(0)) \cdot (F^{-1})'(0) = \frac{f'(\alpha)}{F'(\alpha)} \quad \Rightarrow \quad |f'(\alpha)| \leq |F'(\alpha)| \qquad \square$$

SCHWARZOV PRINCIP ZRCALJENJA

TRDITEV: Naj bo D območje v $\mathbb{C}$: $D^* = \{z \in \mathbb{C};\ \bar{z} \in D\}$ in $f \in \mathcal{H}(D)$.
Tedaj velja: $f^*(z) = \overline{f(\bar{z})}$ in $f^* \in \mathcal{H}(D^*)$.

Dokaz:

$$z \in D^*: \quad \lim_{h \to 0} \frac{f^*(z+h) - f^*(z)}{h} = \lim_{h \to 0} \frac{\overline{f(\bar{z}+\bar{h})} - \overline{f(\bar{z})}}{h} = \lim_{h \to 0} \overline{\left(\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}\right)} = \overline{f'(\bar{z})}$$

$$\alpha \in D;\ \bar{\alpha} \in D^*: \quad f(z) = \sum_0^\infty a_n (z-\alpha)^n$$

$$\overline{f(\bar{z})} = \sum_0^\infty \bar{a}_n (z-\bar{\alpha})^n \quad \leftarrow \text{holomorfna v okolici } \bar{\alpha} \qquad \square$$

Zgled:

$f(z) = z + i$

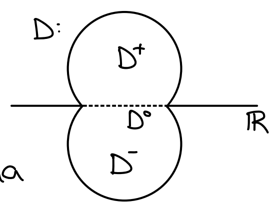
$f^*(z) = \overline{f(\bar{z})} = \overline{\bar{z} + i} = z - i$

$f(z) = z^2$

$f^*(z) = z^2$

IZREK (Schwarzov princip zrcaljenja):

Naj bo D območje v $\mathbb{C}$, simetrično glede na $\mathbb{R}$: $D^* = D$.
Naj bo $D^+ = \{z \in D;\ \operatorname{Im} z > 0\}$; $D^0 = \{z \in D;\ \operatorname{Im} z = 0\}$ in $D^- = \{z \in D;\ \operatorname{Im} z < 0\}$.
Naj bo f zvezna na $D^+ \cup D^0$; holomorfna na D^+ in realna na D^0.
Tedaj obstaja $F \in \mathcal{H}(D)$ taka, da je $F|_{D^+} = f$.



Dokaz:

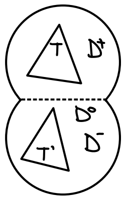
$$F(z) = \begin{cases} f(z); & z \in D^+ \cup D^0 \\ \overline{f(\bar{z})}; & z \in D^- \cup D^0 \end{cases}$$

$$z \in D^0 \to z = \bar{z}$$

$$\overline{f(\bar{z})} = \overline{f(z)} = f(z)$$

F je dobro definirana na D in zvezna na D.
$F \in \mathcal{H}(D^+ \cup D^-)$

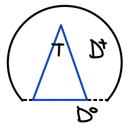
Naj bo $T \subseteq D$ trikotnik $\Rightarrow \int_{\partial T} F(z)dz = 0 \overset{\text{Morer}}{\Rightarrow} F \in \mathcal{H}(D)$.



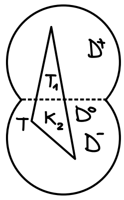
$T \subseteq D^+$: $F = f \rightarrow \int_{\partial T} F(z)dz = 0$ (Cauchy za f)

$T' \subseteq D^-$: $F = f^* \rightarrow \int_{\partial T'} F(z)dz = 0$ (Cauchy za f^*)

$T \subseteq D^+ \cup D^\circ$: $T_\varepsilon \subseteq T \quad \xrightarrow{\varepsilon \searrow 0} \quad T_\varepsilon \rightarrow T$
Eno od stranic T pomaknemo za $\varepsilon > 0$ v D^+.



$$0 = \int_{\partial T_\varepsilon} F(z)dz \xrightarrow[\varepsilon \searrow 0]{} \int_{\partial T} F(z)dz$$



Notranjost T seka D°: $T = T_1 \cup K_2$
$T_1 \subseteq D^+ \cup D^\circ$, $K_2 \subseteq D^- \cup D^\circ$

Kot prej uporabimo Cauchyjev izrek za f na T_1 in f^* na K_2 in upoštevajoč zveznost f na D° in dobimo:
$$\int_{\partial T_1} f(z)dz = 0 \text{ in } \int_{\partial K_2} f^*(z)dz = 0 \Rightarrow \int_{\partial T} F(z)dz = 0$$
$\square$

Zgled:
$f, g \in \mathcal{H}(\mathbb{C})$ celi funkciji.
Katere funkcije f in g ustrezajo enačbi $f^3 + g^3 = 1$.

$$1 = f^3 + g^3 = (f - (-1)g)(f - \alpha g)(f - \bar{\alpha} g) \rightarrow \begin{array}{l} f - (-1)g \neq 0 \\ f - \alpha g \neq 0 \\ f - \bar{\alpha} g \neq 0 \end{array}$$

Imamo več možnosti:
(1) $g \equiv 0 \Rightarrow f^3 = 1 \rightarrow f$ konstantna funkcija: $1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$
(2) $g \not\equiv 0 \Rightarrow \frac{f}{g}$ meromorfna na $\mathbb{C}$, ne zavzame $-1, \alpha, \bar{\alpha}$ $\Big\} \Rightarrow f, g$ konstantni funkciji

f holomorfna na $\mathbb{CP}^1$ je konstantna
meromorfne funkcije na $\mathbb{CP}^1$: končno mnogo polov: $\alpha_1, \dots, \alpha_N$ različni poli (lahko $\alpha_N = \infty$)
α_j: glavni del: $\frac{a_{-N}^j}{(z-\alpha_j)^N} + \dots + \frac{a_{-1}^j}{z - \alpha_j} = P_j\left(\frac{1}{z - \alpha_j}\right)$; $P_j(0) = 0$
$\forall \infty$: $P_N(z)$ polinom: $P_N(0) = 0$
$f(z) - \sum_{j=1}^{N-1} P_j\left(\frac{1}{z-\alpha_j}\right) - P_N(z)$ holomorfna na $\mathbb{CP}^1$ je konstantna
racionalna funkcija: $f(z) = \sum_{1}^{N-1} P_j\left(\frac{1}{z-\alpha_j}\right) - P_N(z) + C$

Zgled:
$f \in \mathcal{H}(\mathbb{C})$: $|f(z)| \leq e^{\operatorname{Re} z} \quad \forall z$
$= |e^z|$
$|e^{-z} f(z)| \leq 1$ $\qquad |\lambda| \leq 1$
$\shortparallel$ $\qquad e^{-z} f(z) = \lambda$
Liouvillov izrek $\qquad f(z) = \lambda e^z$

$f, F \in \mathcal{H}(\mathbb{C})$: $|f(z)| \leq |F(z)| \quad \forall z$
$F(\alpha) \neq 0 \Rightarrow f(z) = \lambda F(z)$; $|\lambda| \leq 1$
$F(\alpha) = 0 \Rightarrow f(\alpha) = 0$: i) $F \equiv 0 \Rightarrow f \equiv 0$
ii) $F \not\equiv 0$: F ima izolirane ničle
$F(z) = (z - \alpha)^N g(z)$; $g \neq 0$ v okolici α
$f \not\equiv 0$: $f(z) = (z-\alpha)^M h(z)$; $h \neq 0$ v okolici α
$\downarrow$

$$|z-\alpha|^M |h(z)| \leq |z-\alpha|^N |g(z)|$$

Ali je $N > M$?

$$|h(z)| \leq |z-\alpha|^{N-M} |g(z)|$$

$h(\alpha) \neq 0$ ničla v α $\Rightarrow M \geq N$

$\Rightarrow \frac{f}{F} \in \mathcal{H}(\mathbb{C})$ in omejena: $f = \lambda F$

dobro definirana v okolici α.

Zgled:

$p(z) = z^6 - 3z^5 - z^2 + 1$ ima 5 ničel na Δ.

na krožnici imamo strogo neenakost

$|z| = 1$: $|z^6 - z^2 + 1| \leq |z|^6 + |z^2| + 1 = 3 \Leftrightarrow z^6 \geq 0$

$-z^2 \geq 0 \Rightarrow -z^6 \geq 0$

Zgled:

$f(z) = \frac{z}{1+az+z^2}$, $a \in \mathbb{R} \cap [-2,2]$

f je univalentna (injektivna) na Δ

poli: $z^2 + az + 1 = 0$

$z_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2-4}) = \frac{1}{2}(-a \pm i\sqrt{4-a^2})$

$|z_{1,2}|^2 = \frac{1}{4}(a^2 + 4 - a^2) = 1$

injektivnost: $\frac{z}{1+az+z^2} = \frac{w}{1+aw+w^2} \Leftrightarrow z + awz + zw^2 = w + awz + wz^2$

$|z| < 1$; $|w| < 1$

$z - w = zw(z-w)$

$\Rightarrow wz = 1 \Rightarrow z \neq w$

$f(\partial\Delta)$: $f(z) = \frac{z}{1+az+z^2} = \frac{1}{\bar{z}+a+z} = \frac{1}{2\mathrm{Re}\,z + a}$

$|z| = 1$

$z\bar{z} = 1$

$-1 \leq \mathrm{Re}\,z \leq 1$

$a = 1$: $x \mapsto \frac{1}{2x+1}$; $-1 \leq x \leq 1$